Post-Newtonian approximation *and gravitational waves from compact binaries I.*

Yousuke Itoh (伊藤 洋介)
KAGRA Gravitational Wave Data Analysis
International cooperation section
Research Center for the Early Universe,
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@TGW mini-School, National Taiwan Normal University, Taipei, 13-15 July 2017

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Self introduction:

- Member of the KAGRA & LIGO Scientific collaboration.
  - But this talk is from my personal view ….
- Ph.D thesis on post-Newtonian equations of motion for relativisitic compact binaries. [Supervisor: Toshifumi Futamase@Tohoku Univ., 2002]
- Work as a postdoc at Albert-Einstein Institute at Potsdam, Univ. Wisconsin-Milwaukee, & Tohoku Univ.
- Current research field: data analysis on GWs from pulsars

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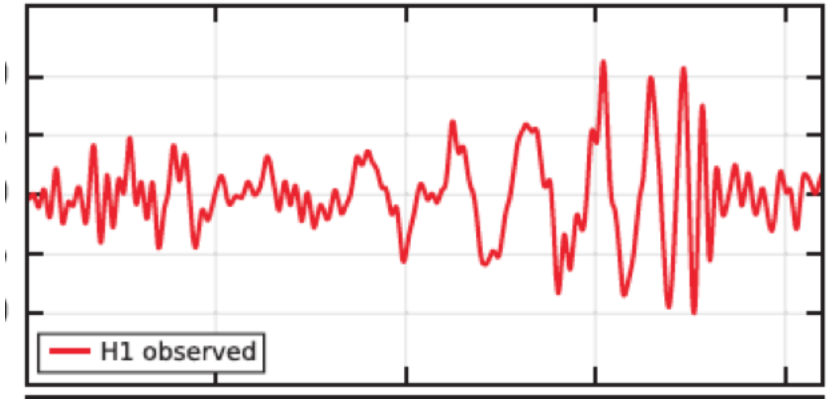
3. Newtonian waveform calculation
4. PN equations of motion
5. PN waveform

INTRODUCTION

Data analysis

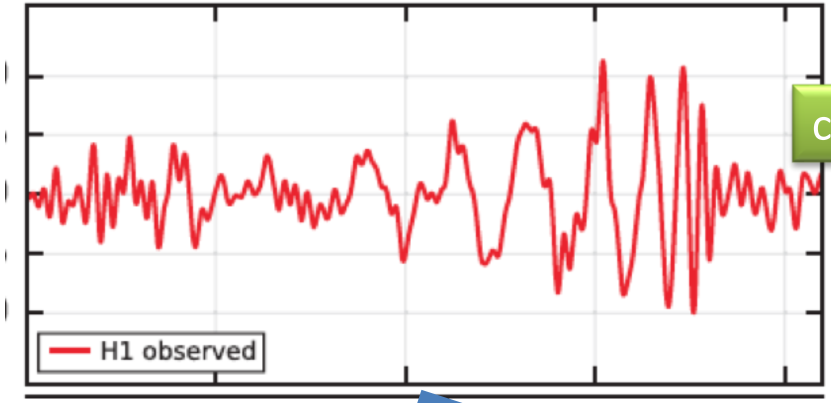
Observed GW



Want to extract physical information

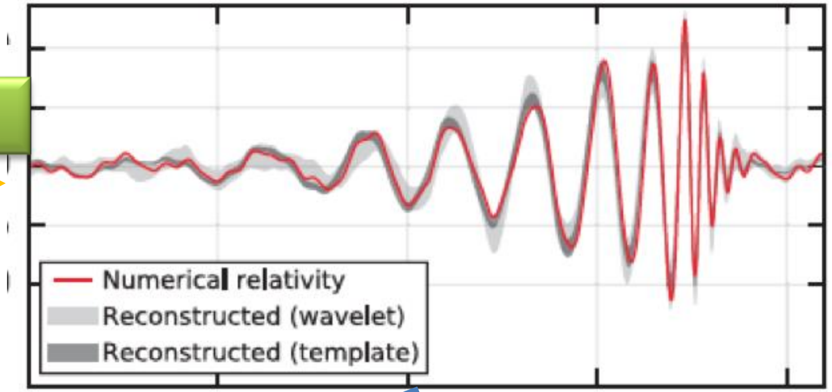
Data analysis

Observed GW



compare

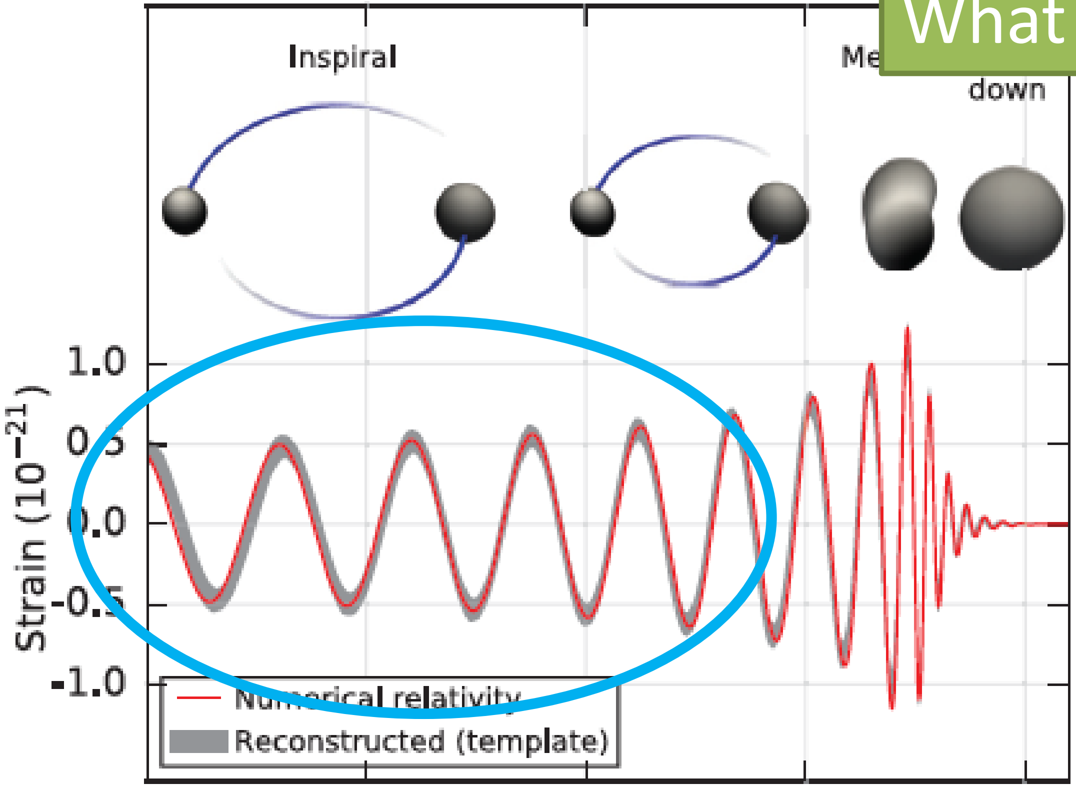
Theoretical expectations



$$< s|h > (t) = 4\mathrm{Re}\int_0^\infty \frac{\tilde{s}(f)\tilde{h}^*(f)}{S_h(f)} e^{2\pi i f t} df$$

Want to extract physical information
Compute correlation with theoretical expectation
Find the model that maximizes the correlation

What we can learn

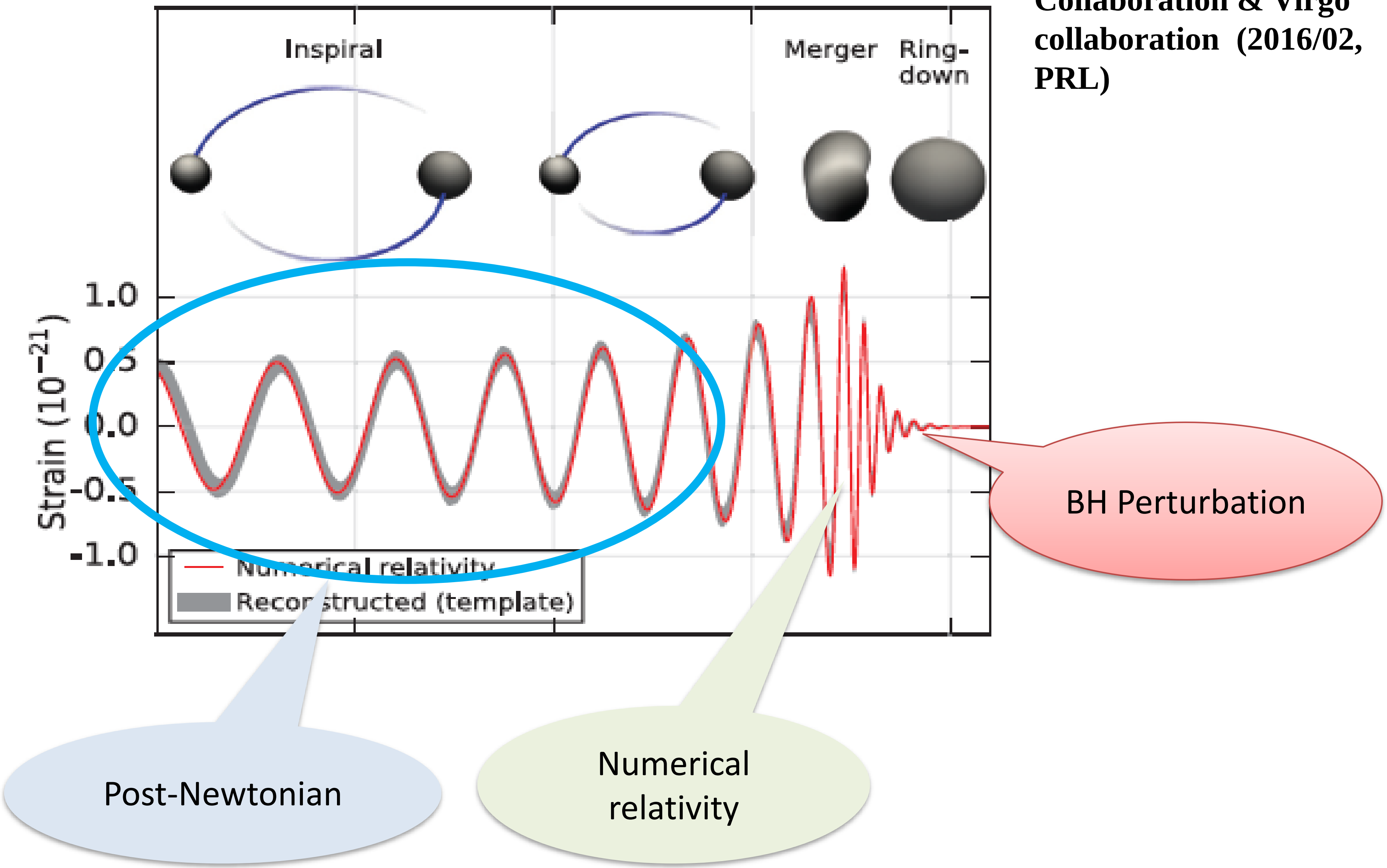


$$h(t) = \frac{G\mathcal{M}}{c^2 r} U(\vec{n}, \iota, \psi) \left(\frac{5G\mathcal{M}/c^3}{t_c - t}\right)^{1/4} \cos\left(\Phi_0 - \left(\frac{t_c - t}{5G\mathcal{M}/c^3}\right)^{5/8} + (\mathrm{BH\,spins})\right)$$

Luminosity distance, sky location $\vec{n}$, masses, spin angular momentums of the black holes of the binary

PNA, Numerical relativity, Single star/BH Perturbation

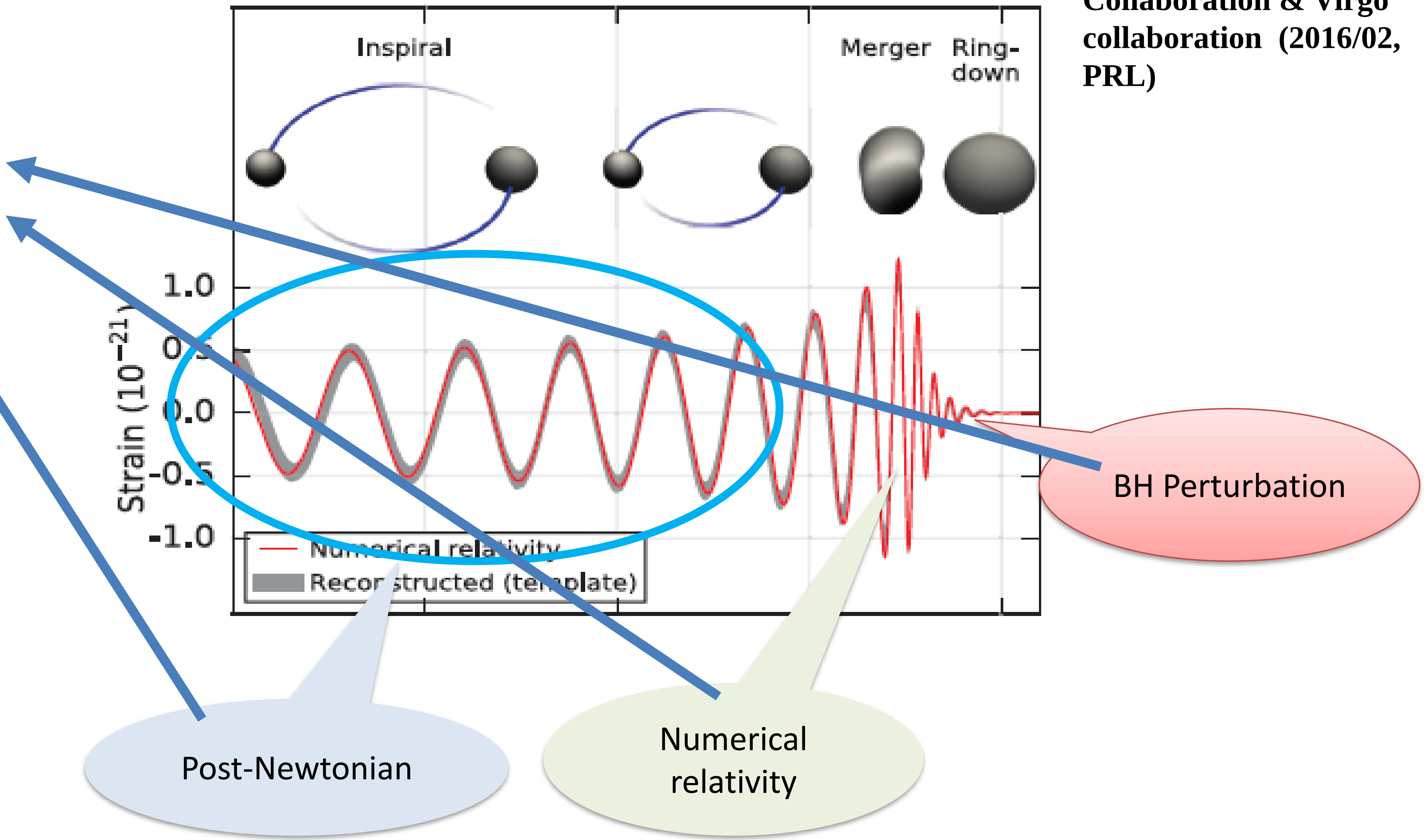
LIGO Scientific Collaboration & Virgo collaboration (2016/02, PRL)



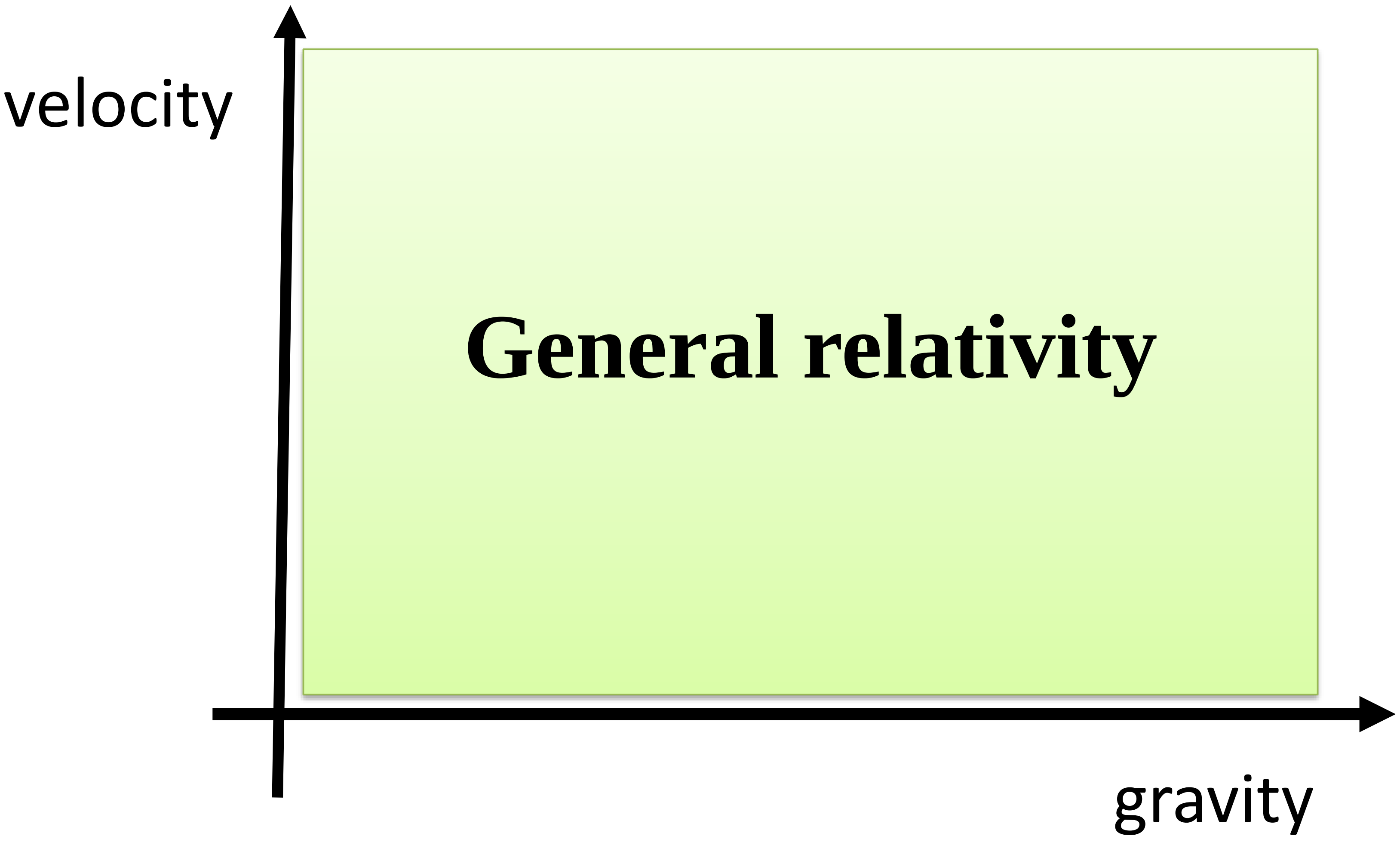
PNA, Numerical relativity, Single star/BH Perturbation

LIGO Scientific Collaboration & Virgo collaboration (2016/02, PRL)

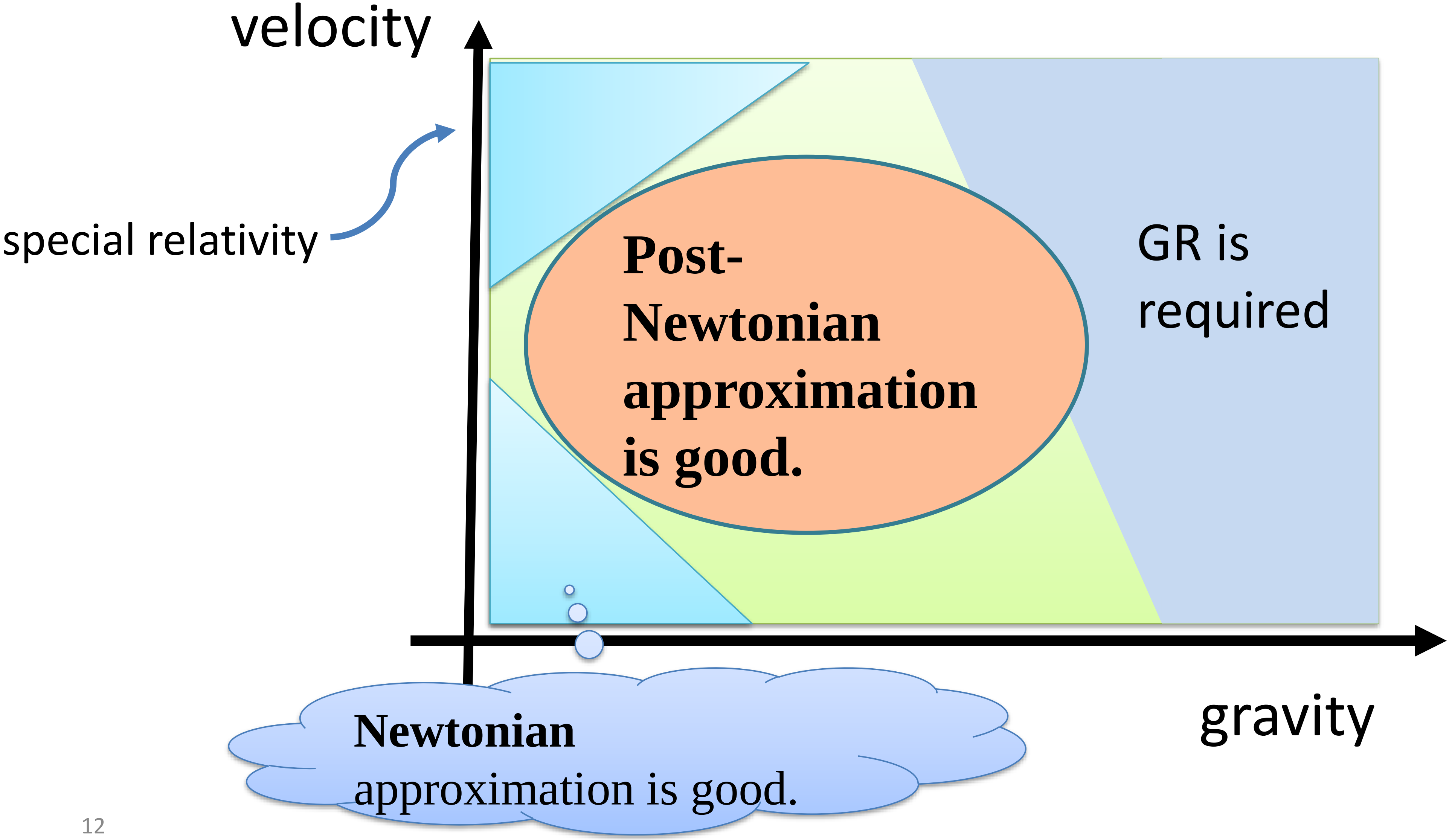
Combined all three
- ✓ SEOBNRv4
- ✓ IMRPhenomP



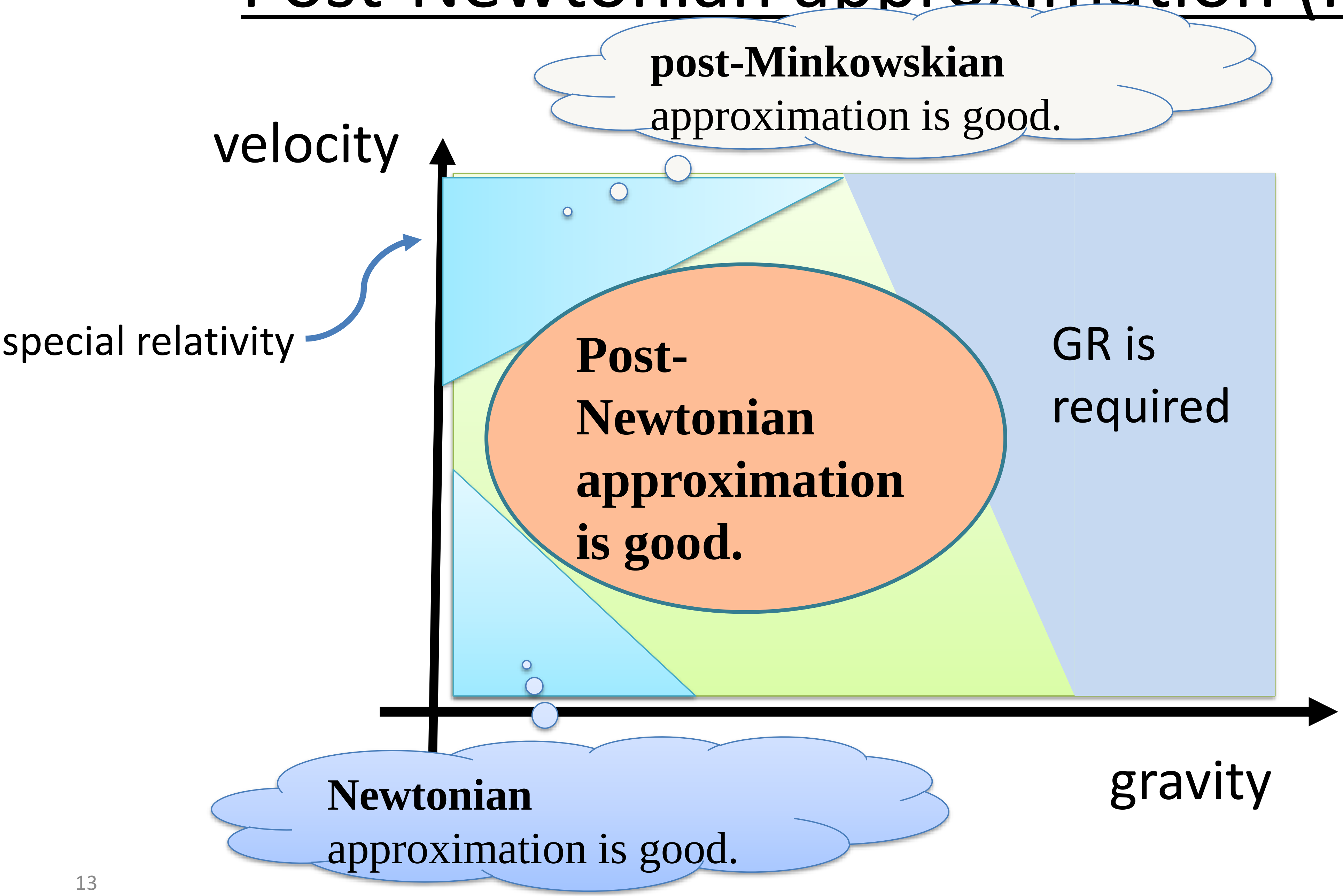
If we are "perfect computers".



Need Post-Newtonian approximation (PNA)



Post-Newtonian approximation (PNA)



Post-Minkowskian approximation (PMA)

- Expansion parameters of PMA:

  ~~velocity: v/c~~

  gravity: $GM/(c^2R) = \boldsymbol{\lambda}$
- It is in essence a weak field approximation.
- Expand the Einstein equations and equations of conservation law of matter's stress energy tensor.

$$G[g(x;\lambda)] = T(\lambda m, v, \cdots) \qquad \nabla_{[g(x;\lambda)]}T(\lambda m, v, \cdots) = 0$$

and metric: $g(x;\epsilon) = \eta_M + G g_1(x) + G^2 g_2(x) + \cdots$

v, M, R: typical velocity, mass, and length of the system)

Post-Newtonian approximation (PNA)

- Expansion parameters of PNA:

  velocity: v/c

  gravity: $GM/(c^2R)$

- Assume it is approximately Newtonian bounded system:

$$\boxed{\mathcal{O}\left(\frac{v^2}{c^2}\right) = \mathcal{O}\left(\frac{GM}{c^2R}\right) \equiv \epsilon^2}$$

- Expand the Einstein equations and equations of conservation law of matter's stress energy tensor in ε.

$$G[g(x,\epsilon)] = T(x;\epsilon^2 m, \epsilon v, \cdots) \qquad \nabla_{[g(x;\epsilon)]} T(x;\epsilon^2 m, \epsilon v, \cdots) = 0$$

- Solve the equations order by order.

v, M, R: typical velocity, mass, and length of the system)

Post-Newtonian approximation (PNA)

- Expand the Einstein equations and equations of conservation law of matter's stress energy tensor.

$$G[g(x;\epsilon)] = T(\epsilon^2 m, \epsilon v, \cdots) \qquad \nabla_{[g(x;\epsilon)]} T(\epsilon^2 m, \epsilon v, \cdots) = 0$$

- Solve the equations order by order. Then the metric would be obtained as a polynomial functional series in ε.

$$g(x;\epsilon) = \eta_N + \epsilon^2 g_1(x) + \epsilon^4 g_2(x) + \cdots$$

Post-Newtonian approximation (PNA)

- Equations of motion for a two point-particle system would be obtained as

$$\frac{dv_1^i}{dt} = -\frac{Gm_1m_2}{r_{12}^3}r_{12}^i + \epsilon^2 F_{1PN}^i + \epsilon^4 F_{2PN}^i + \epsilon^5 F_{2.5PN}^i + \epsilon^6 F_{3PN}^i + \cdots$$

- ε^n correction to the lowest order term is called (n/2) PN correction.
- Since terms multiplied by ε to the power of odd-integers are time non-reversible, they represent energy dissipation. The radiation reaction term first appears at 2.5PN order [cf. v^3 term in EM case.].
- But there appears a radiation reaction force at the 4 PN order in EOM due to a tail effect.

Post-Newtonian approximation (PNA)

- Equations of motion for a two point-particle system is obtained

$$\frac{dv_1^i}{dt} = -\frac{Gm_1m_2}{r_{12}^3} r_{12}^i + \epsilon^2 F_{1PN}^i + \epsilon^4 F_{2PN}^i + \epsilon^5 F_{2.5PN}^i + \epsilon^6 F_{3PN}^i + \cdots$$

- There is a conservative Energy when radiation reaction effects are neglected.

$$E = -\frac{Gm_1m_2}{2r_{12}} + \epsilon^2 E_{1PN} + \epsilon^4 E_{2PN} + \epsilon^6 E_{3PN} + \cdots$$

- Similarly we have GW luminosity $L = L_0 + \epsilon^2 L_1 + \cdots$

Post-Newtonian approximation (PNA)

- While it is true that L_0 corresponds to the 2.5 PN radiation reaction effect in EOM, we say L_1 the 1 PN correction in GW luminosity to the leading order term (L_0).

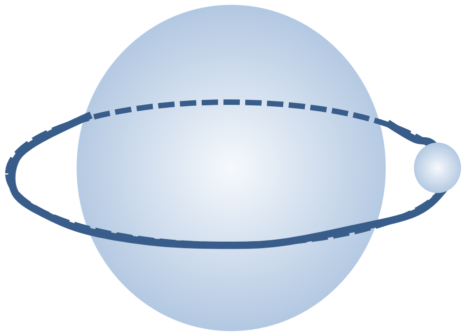
$$\frac{d}{dt}\left(E_N + \epsilon^2 E_1 + \cdots\right) = L_0 + \epsilon^2 L_1 + \cdots$$

or

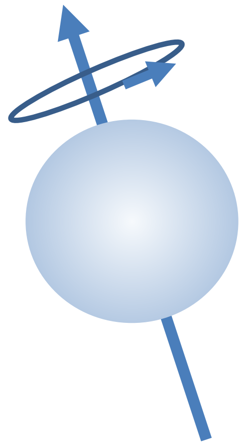
$$\frac{dE_N}{dt} = L_0 + \mathcal{O}\left(\epsilon^2\right)$$

Other useful expansions

- extreme mass ratio limit
  - small stellar object orbiting around a super-massive black hole [eLISA target].
  - self-force approach [e.g., E. Poisson, Liv. Rev. Rel.]
  - the lowest order EOM is a geodesic equation.



- multipole expansion of a stellar object
  - good approximation for compact objects like neutron stars & black holes.
  - when tidal effects can be neglected (good in the inspiralling phase), mass and spin are enough.



Mass-Spin approximation

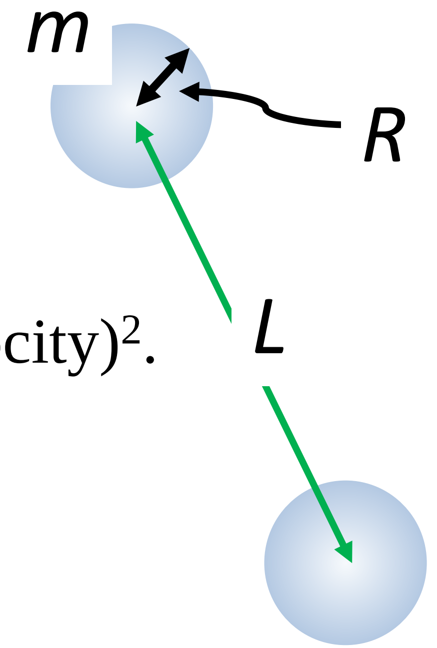
Rough argument: (m, v_s, R: mass, spinning velocity, and radius of a star, L: orbital separation)

1. Tidal gravity force: $m R / L^3$

Tidally induced quadrupole: $Q \sim$ (tidal gravity)/(self gravity) times $m R^2 \sim m^3 (R/L)^3$.

Quadrupole orbit coupling force: $F \sim mQ/L^4 \sim (m/L)^7 = 5$ PN

(cf. $(m/L)^2$ for Newtonian Force. For a compact star $R \sim m$).



2. Spin induced quadrupole: $Q \sim (mRv)^2/m \sim m^3 v^2$.

Quadrupole orbit coupling force: $F \sim (m/L)^4 v^2 = 2$PN times (rotational velocity)^2.

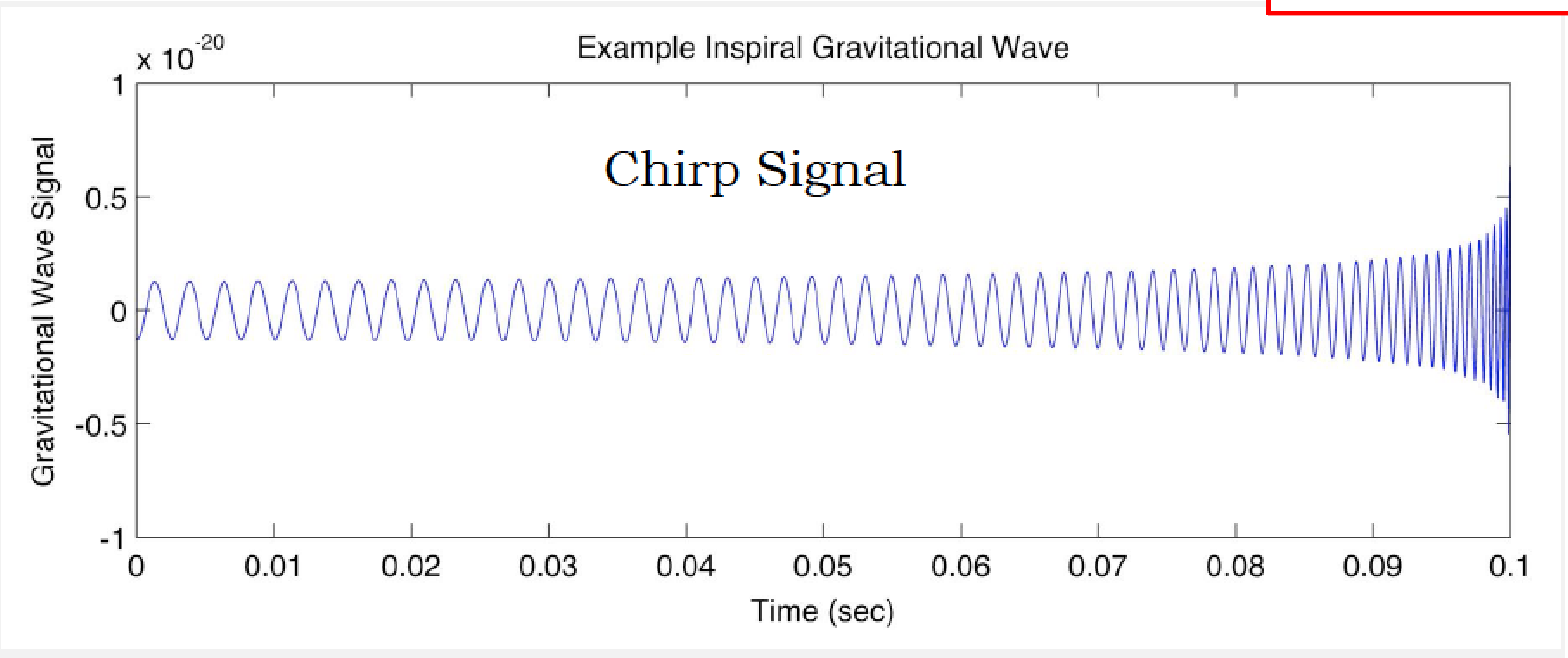
See e.g. for Bildsten & Cutler (1992), Blanchet's 2007 Liv. Rev. review.

NEWTONIAN WAVEFORM CALCULATION

Let’s compute GWs from a point particle binary in an inpiraling phase

$$h_{ij}^{TT} = \frac{2G\ddot{I}_{ij}^{TT}}{c^4 r} + \cdots$$

Want to draw the following curve.



Let's compute GWs from a point particle binary

$$\vec{x}_1(t) = \frac{m_2}{m_1+m_2} a(\cos(\omega_o t), \sin(\omega_o t), 0),$$
$$\vec{x}_2(t) = -\frac{m_1}{m_1+m_2} a(\cos(\omega_o t), \sin(\omega_o t), 0),$$

$$I_{xx} = \frac{1}{2}\mu a(1+\cos(2\omega_o t)),$$
$$I_{xy} = \frac{1}{2}\mu a \sin(2\omega_o t),$$
$$I_{yy} = \frac{1}{2}\mu a(1-\cos(2\omega_o t)),$$

$$\ddot{I}_{xx} = -2\omega_o^2 \mu a \cos(2\omega_o t),$$
$$\ddot{I}_{xy} = -2\omega_o^2 \mu a \sin(2\omega_o t),$$
$$\ddot{I}_{yy} = 2\omega_o^2 \mu a \cos(2\omega_o t)),$$

Binary with orbital radius a , angular frequency ω_o

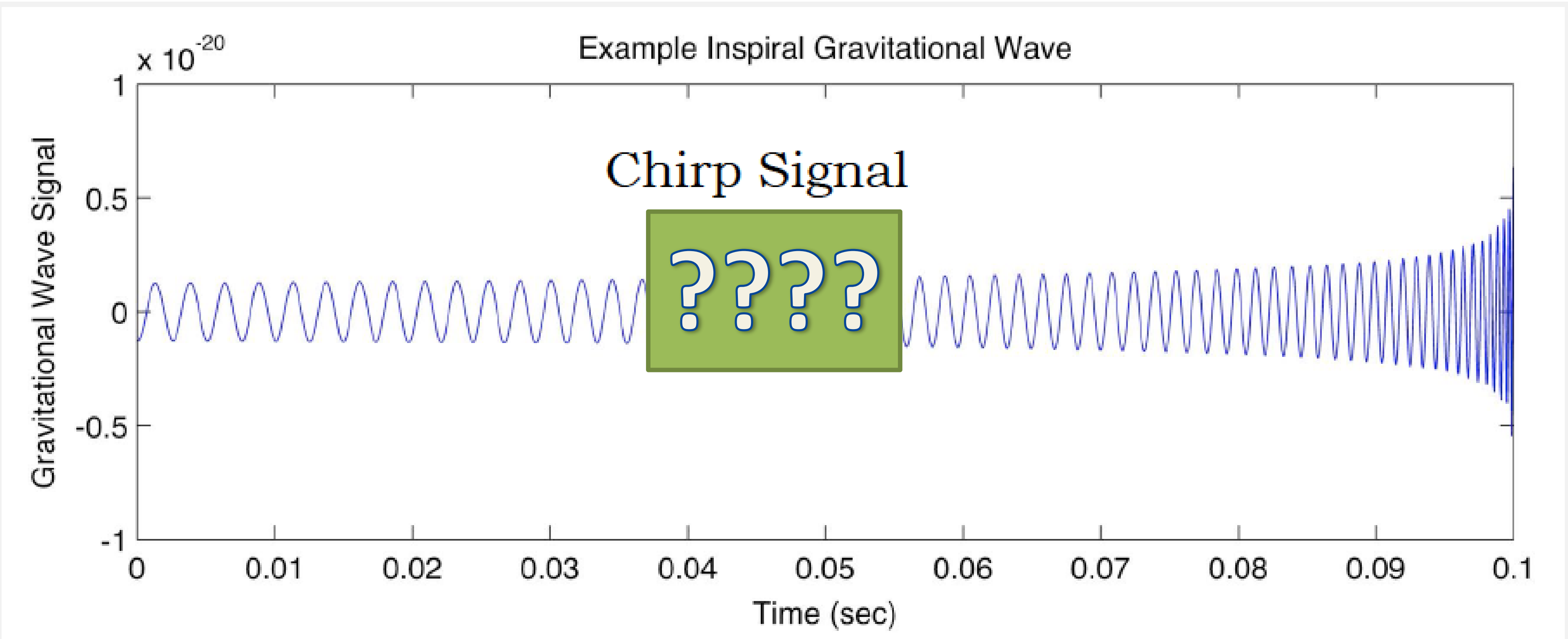
Compute quadrupole moment. Let μ denote reduced mass

$$\mu = m_1 m_2/(m_1+m_2)$$

Take temporal derivatives twice, compute transverse-traceless part.

Let's compute GWs from a point particle binary

$$h_{ij}^{TT} = \frac{2G\ddot{I}_{ij}^{TT}}{c^4 r} = \frac{4G\mu a\omega_o^2}{c^4 r}\begin{pmatrix} -\cos(2\omega_o t) & -\sin(2\omega_o t) & 0 \\ -\sin(2\omega_o t) & \cos(2\omega_o t) & 0 \\ 0 & 0 & 0 \end{pmatrix}$$



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$$h_{ij}^{TT} = \frac{2G\ddot{I}_{ij}^{TT}}{c^4 r} = \frac{4G\mu a\omega_o^2}{c^4 r}\begin{pmatrix} -\cos(2\omega_o t) & -\sin(2\omega_o t) & 0 \\ -\sin(2\omega_o t) & \cos(2\omega_o t) & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

How can we compute time series h(t) (scalar) from tensor $h^{TT}_{ij}(t)$?

Detector output time series

Linear perturbation solution to the Einstein equations

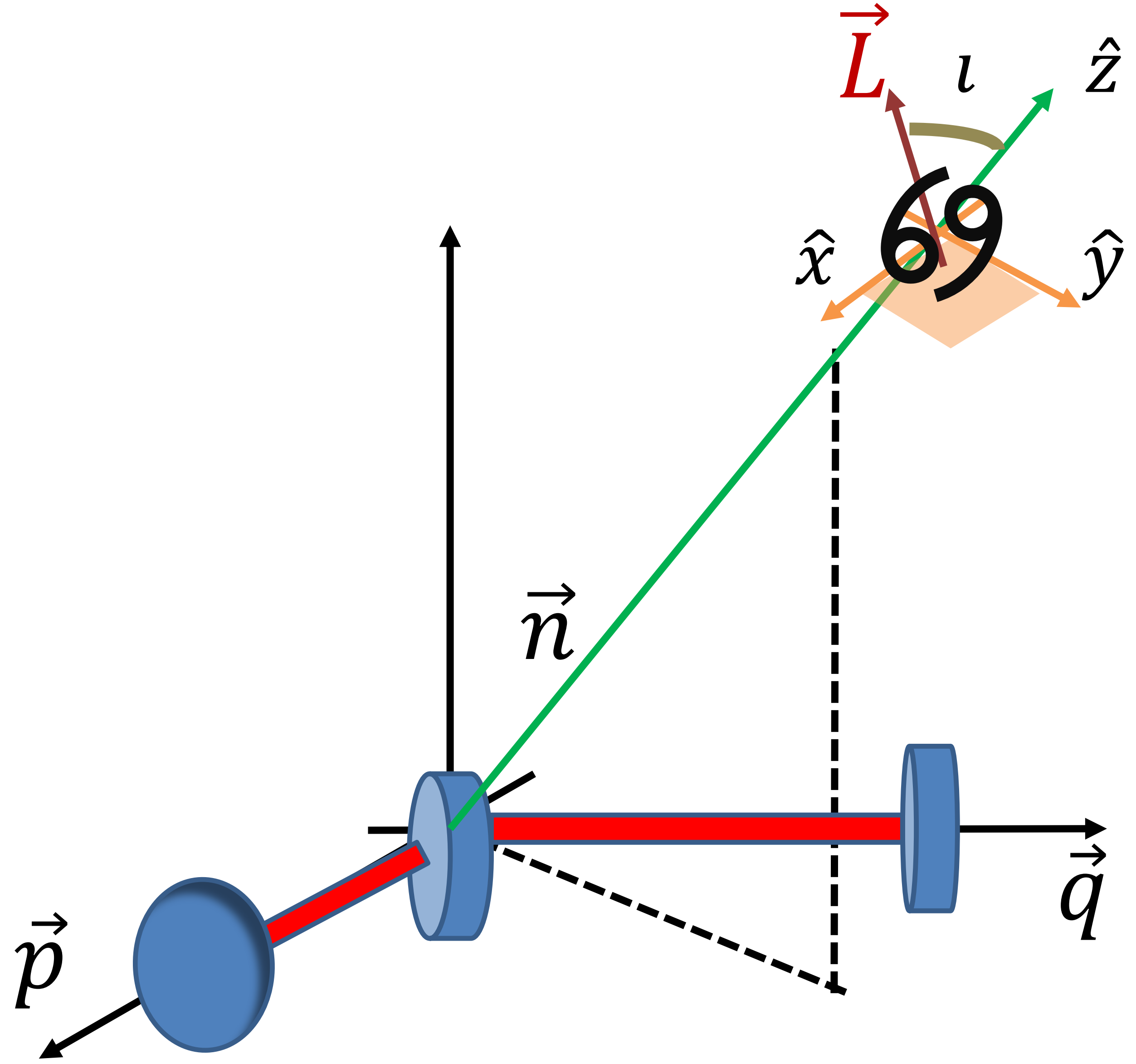
$$h_{ij}^{TT} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_\times & 0 \\ 0 & h_\times & -h_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_{ij} = e_{ij}^{+} h_+ + e_{ij}^{\times} h_\times$$

Polarization tensors

$$e_{ij}^{+} = \hat{x}_i \hat{x}_j - \hat{y}_i \hat{y}_j,$$
$$e_{ij}^{\times} = \hat{y}_i \hat{x}_j + \hat{x}_i \hat{y}_j$$

Detector arm vectors p & q, Antenna pattern function F_+、F_x

$$\begin{aligned} h(t) &= \frac{1}{2}(\hat{p}^i \hat{p}^j - \hat{q}^i \hat{q}^j) h_{ij}^{TT}(t) \\ &= F_+(\vec{n}, \psi) h_+(t) + F_\times(\vec{n}, \psi) h_+(t) \end{aligned}$$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ι
$\hat{z}$
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$\hat{y}$
$\vec{n}$
$\vec{q}$
$\vec{p}$

Detector output time series

Linear perturbation solution to the Einstein equations

$$h_{ij}^{TT} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_\times & 0 \\ 0 & h_\times & -h_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_{ij} = e_{ij}^{+} h_+ + e_{ij}^{\times} h_\times$$

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Detector arm vectors p & q, Antenna pattern function F_+、F_x

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Let's compute GWs from a point particle binary

$$\boxed{L_{\rm gw} = \frac{G}{5c^5}\langle \dddot{I}_{kl}\dddot{I}^{kl}\rangle = \frac{32c^5}{5G}\left(\frac{G\mathcal{M}_c\omega_o}{c^3}\right)^{10/3}}$$

$$\mathcal{M}_c = (m_1 m_2)^{3/5}/(m_1+m_2)^{1/5}$$

Compute GW energy using the quadrupole formula. Mc is a chirp mass

$$\boxed{E_{\rm orbit} = \frac{1}{2}\mu v^2 - \frac{G\mu m_t}{R} = -\frac{\mu c^2}{2}\left(\frac{Gm_t\omega_o}{c^3}\right)^{2/3}}$$

L_{gw} is from the orbital energy.

$$\boxed{dE_{\rm orbit}/dt + L_{\rm gw} = 0}$$

$$\boxed{\dot{\omega}_o = \frac{96}{5}\left(\frac{G\mathcal{M}_c}{c^3}\right)^{5/3}\omega_o^{11/3}}$$

Get a differential equation for ω_o

$$\omega_o(t) = \frac{5^{3/8}}{8}\left(\frac{G\mathcal{M}_c}{c^3}\right)^{-5/8}(t_c - t)^{-3/8},$$

$$\Phi_o(t) = -\int_t^{t_c}\omega_o(t)dt = -\left(\frac{5G\mathcal{M}_c}{c^3(t_c-t)}\right)^{-5/8}$$

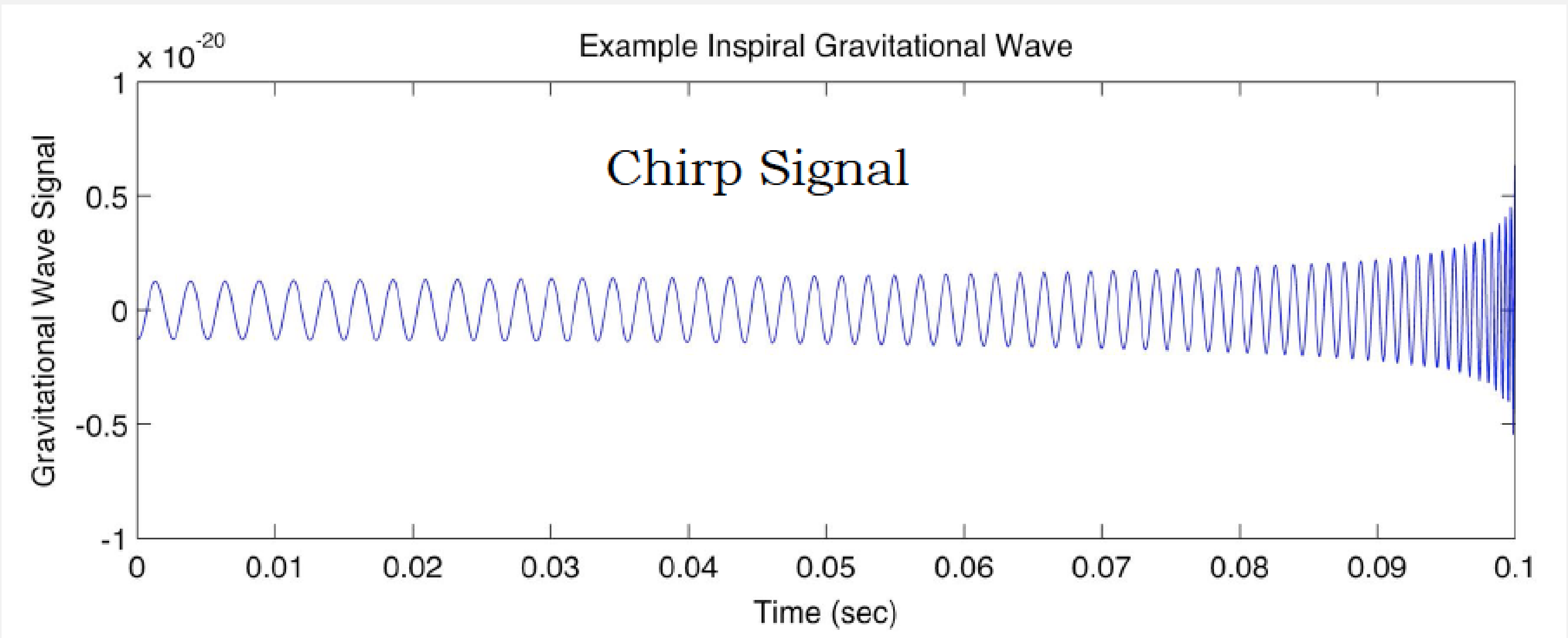
Solve it, find the phase evolution equation.

Let's compute GWs from a point particle binary

$$h_+(t) = -A(t)\frac{1}{2}(1+\cos^2\iota)\cos(2\Phi_o(t)),$$

$$h_\times(t) = -A(t)\cos\iota\sin(2\Phi_o(t)),$$

$$A(t) = \frac{G\mathcal{M}_c}{c^2 r}\left(\frac{5G\mathcal{M}_c}{c^3(t_c-t)}\right)^{1/4}$$



Let's compute GWs from a point particle binary

$$\tilde{h}_+(f_{\rm gw}) = \int_{-\infty}^{\infty} dt e^{-2\pi i f_{\rm gw} t} h_+(t) = \frac{1}{2}\int_{-\infty}^{t_c} dt e^{-2\pi i f_{\rm gw} t} A_+(t) \mathrm{e}^{i\Phi_{\rm gw}(t)}$$

Use stationary phase approximation. Note that the integrand oscillates so rapidly that it amounts to zero for any frequencies other than

$$d\Phi_{\rm gw}(t_f)/dt = f_{\rm gw}$$

$$\begin{aligned}
\tilde{h}_+(f_{\rm gw}) &\simeq \frac{1}{2}\int_{-\eta}^{\eta} dt A_+(t) \exp\left[i\Phi_{\rm gw}(t_f) - 2\pi i f_{\rm gw} t_f + \frac{i}{2}\ddot{\Phi}_{\rm gw}(t_f)(t-t_f)^2\right] \\
&\simeq \frac{1}{2} A_+(t_f) \mathrm{e}^{i\Phi_{\rm gw}(t_f) - 2\pi i f_{\rm gw} t_f} \int_{-\eta}^{\eta} dt \exp\left[\frac{i}{2}\ddot{\Phi}_{\rm gw}(t_f) t^2\right] \\
&\simeq \frac{1}{2}\sqrt{\frac{2\pi}{|\ddot{\Phi}_{\rm gw}(t_f)|}} A_+(t_f) \mathrm{e}^{i\Phi_{\rm gw}(t_f) - 2\pi i f_{\rm gw} t_f + \frac{i\pi}{4}}
\end{aligned}$$

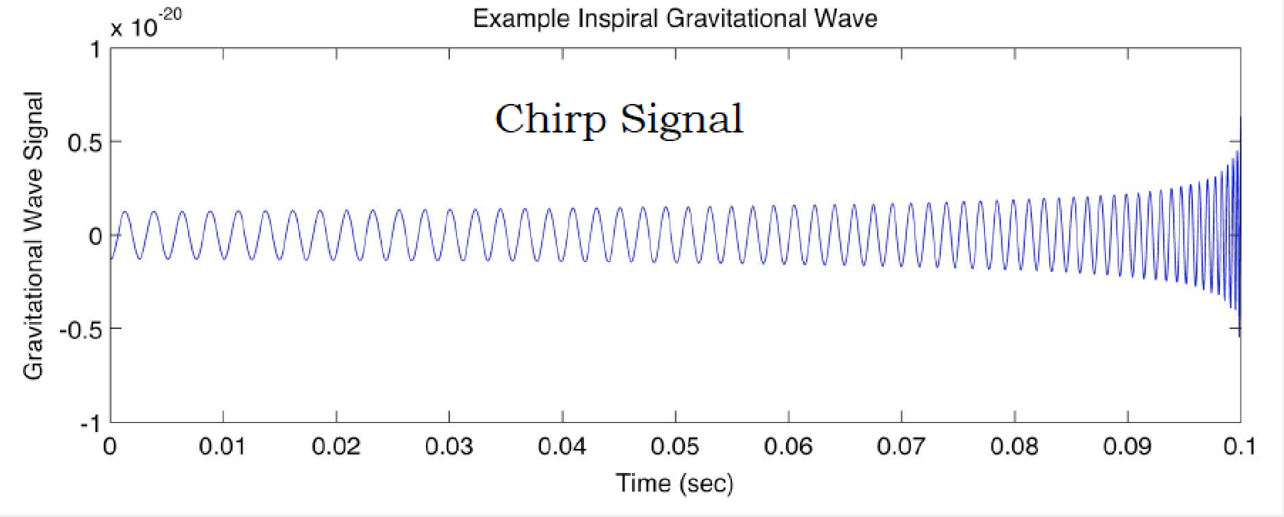
Useful quantities

Time to coalescence from f_{gw} Hz

$$\tau_c = \frac{5}{256}\left(\frac{\pi G\mathcal{M}_c f_{gw}}{c^3}\right)^{-8/3}\left(\frac{G\mathcal{M}_c}{c^3}\right) \simeq 3\,\mathrm{mins}\left(\frac{f_{gw}}{19\mathrm{Hz}}\right)^{-8/3}\left(\frac{\mathcal{M}_c}{1.2M_\odot}\right)^{-5/3}$$

Frequency as a function of the orbital radius

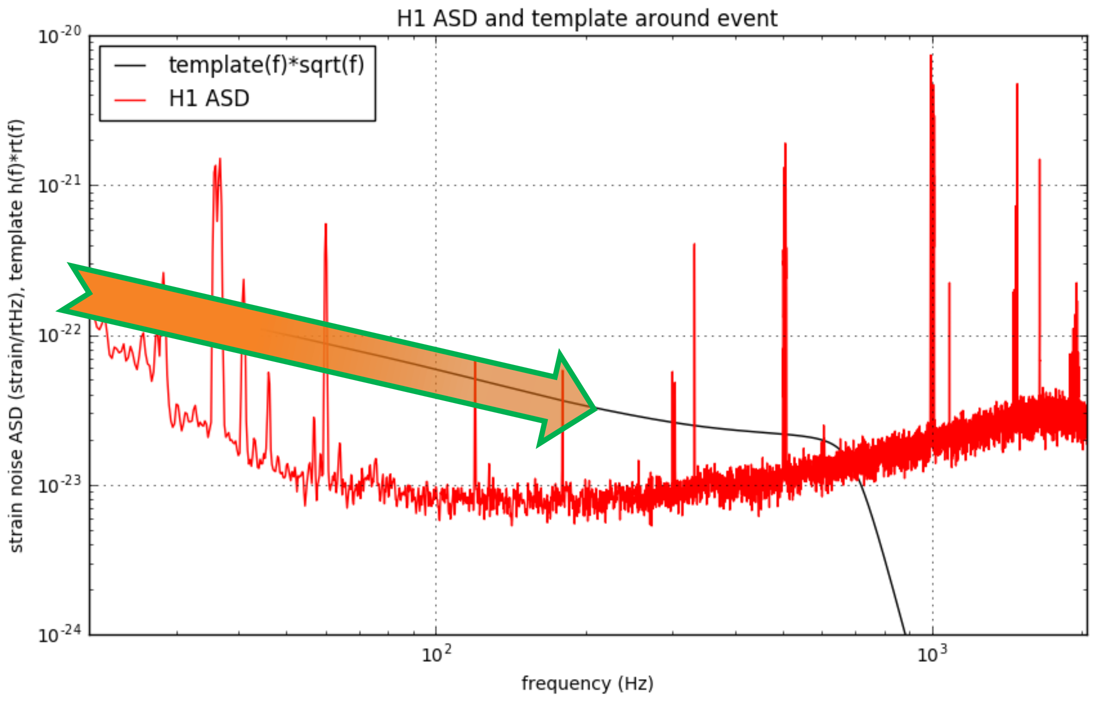
$$f_{\mathrm{gw}} = \frac{1}{\pi}\left(\frac{Gm_t}{a^3}\right)^{1/2} \simeq 19\mathrm{Hz}\left(\frac{m_t}{2.8M_\odot}\right)^{1/2}\left(\frac{a}{472\mathrm{km}}\right)^{-3/2}$$



$$h_+(t) = -A(t)\frac{1}{2}(1+\cos^2\iota)\cos(2\Phi_o(t)),$$

$$h_\times(t) = -A(t)\cos\iota\sin(2\Phi_o(t)),$$

$$A(t) = \frac{G\mathcal{M}_c}{c^2 r}\left(\frac{5G\mathcal{M}_c}{c^3(t_c - t)}\right)^{1/4}$$



$$\tilde{h}_+(f) = -A(f)\frac{1}{2}(1+\cos^2\iota)\exp\left[-i\Psi(f) - 2i\phi\right],$$

$$\tilde{h}_\times(f) = -A(f)\cos\iota\exp\left[-i\Psi(f) - \frac{i\pi}{2} - 2i\phi\right],$$

$$A(f) \equiv \left(\frac{5\pi}{24}\right)^{1/2}\frac{c}{r}\left(\frac{G\mathcal{M}_c}{c^3}\right)^2\left(\frac{\pi G\mathcal{M}_c f}{c^3}\right)^{-7/6},$$

$$\Psi(f) \equiv 2\pi f t_c - \frac{\pi}{4} - \phi_c + \frac{3}{128}\left(\frac{\pi G\mathcal{M}_c f}{c^3}\right)^{-5/3}$$

$$\text{Power} = \int S_h(f)\, d\ln f,$$

$$S_h^{1/2}(f) = \sqrt{f|\tilde{h}(f)|^2} \propto f^{-2/3}$$

recapitulate: making waveform

Derive the far zone metric.
It depends on source multipole moments (SMM).

Derive PN EOM and evaluate the SMM.

Derive orbital energy directly or from PN EOM.

Derive GW luminosity from the far zone metricand solve balance equation to derive GW phase evolution.

Combine all, we obtain far zone field (what an observer measures) in terms of the physical quantities of the system.

$$h_{ij}^{TT} = \frac{2G\ddot{I}_{ij}^{TT}}{c^4 r} + \cdots$$

$$\ddot{I}_{ij}^{TT}(m_1, m_2, \Omega, \cdots), \cdots \leftarrow \text{(circular) EOM}$$

$$E(m_1, m_2, \Omega, \cdots), \cdots \leftarrow \text{(circular) EOM}$$

$$\frac{dE(m_1, m_2, \Omega, \cdots)}{dt} = L(m_1, m_2, \Omega, \cdots) \rightarrow \Omega = \Omega(t)$$

$$h_{ij}^{TT} = \frac{2G\ddot{I}_{ij}^{TT}(m_1, m_2, \Omega(t))}{c^4 r} + \cdots$$

How accurately do we need to know the waveform?

$$< s|h > (t) = 4\mathrm{Re}\int_0^\infty \frac{\tilde{s}(f)\tilde{h}^*(f)}{S_h(f)} e^{2\pi i f t} df$$

To have large cross-correlation, we do not want to miss one cycle among 10^4 cycles in the detector band (NS/NS). Assume v ~ 0.3c then $(v/c)^n \sim 10^{-4}$ or n = 7.6. Hence, 3 ~ 4 PN corrections in the waveform would be necessary.

Number of cycles from each PN correction

PN order		$1.4\,M_\odot + 1.4\,M_\odot$	$10\,M_\odot + 1.4\,M_\odot$	$10\,M_\odot + 10\,M_\odot$
N	(inst)	15952.6	3558.9	598.8
1PN	(inst)	439.5	212.4	59.1
1.5PN	(leading tail)	–210.3	–180.9	–51.2
2PN	(inst)	9.9	9.8	4.0
2.5PN	(1PN tail)	–11.7	–20.0	–7.1
3PN	(inst + tail-of-tail)	2.6	2.3	2.2
3.5PN	(2PN tail)	–0.9	–1.8	–0.8

PN order		$1.4\,M_\odot + 1.4\,M_\odot$	$10\,M_\odot + 1.4\,M_\odot$	$10\,M_\odot + 10\,M_\odot$
1.5PN	(leading SO)	$65.6\kappa_1\chi_1 + 65.6\kappa_2\chi_2$	$114.0\kappa_1\chi_1 + 11.7\kappa_2\chi_2$	$16.0\kappa_1\chi_1 + 16.0\kappa_2\chi_2$
2.5PN	(1PN SO)	$9.3\kappa_1\chi_1 + 9.3\kappa_2\chi_2$	$33.8\kappa_1\chi_1 + 2.9\kappa_2\chi_2$	$5.7\kappa_1\chi_1 + 5.7\kappa_2\chi_2$
3PN	(leading SO-tail)	$-3.2\kappa_1\chi_1 - 3.2\kappa_2\chi_2$	$-13.2\kappa_1\chi_1 - 1.3\kappa_2\chi_2$	$-2.6\kappa_1\chi_1 - 2.6\kappa_2\chi_2$
3.5PN	(2PN SO)	$1.9\kappa_1\chi_1 + 1.9\kappa_2\chi_2$	$11.1\kappa_1\chi_1 + 0.8\kappa_2\chi_2$	$1.7\kappa_1\chi_1 + 1.7\kappa_2\chi_2$
4PN	(1PN SO-tail)	$-1.5\kappa_1\chi_1 - 1.5\kappa_2\chi_2$	$-8.0\kappa_1\chi_1 - 0.7\kappa_2\chi_2$	$-1.5\kappa_1\chi_1 - 1.5\kappa_2\chi_2$

$\kappa_a = \hat{S}_a \cdot \boldsymbol{\ell}$

Integrated from 10 Hz to ISCO: $1/(6^{3/2}\,\boldsymbol{\pi}(m_1+m_2))$
The table is from the Blanchet's LRR review.

POST-NEWTONIAN EQUATIONS OF MOTION

Waveform Templates and Equations of Motion

$$h(t) \sim \frac{G^2 Q m_1 m_2}{c^4 R r_{12}(t)} \cos\left(\int 2\pi f(t) dt\right),$$

Need to know Phase evolution

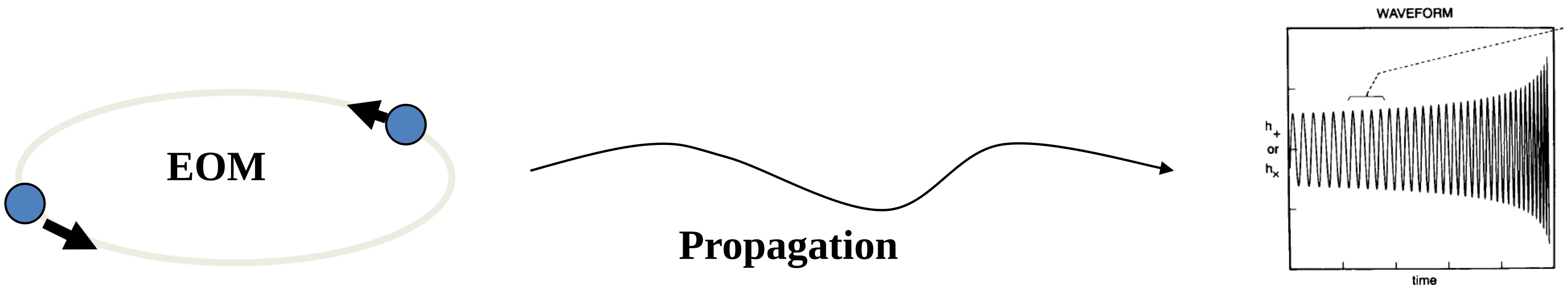
EOM → orbital evolution → GW Phase evolution

$$\mu a^i = -\frac{G(m_1+m_2)}{r_{12}^3} r_{12}^i + \cdots; \quad \frac{df}{dt} = \frac{96\pi^{8/3} G^{5/3}}{5c^5} M_{\rm chirp}^{5/3} f^{11/3} + \ldots,$$

- **More accurate the EOM is, better the quality of waveform templates becomes and we get good signal to noise ratio.**
- **For GW detection and measurements, 3.5 (~ 4 PN) EOM may be enough for stellar mass binary.**

EOM and wave propagation problem.

- Both EOM and wave propagation from source to observer must be computed to construct waveform.
  - Blanchet-Damour-Iyer (BDI) et al or Will-Wiseman (WW) succeeded in deriving higher order waveform.
  - This talk is on EOM.



Two approaches to PNA EOM.

- Two approaches to find PNA binary dynamics in insipiralling phase.
  1. ADS Hamiltonian in ADMTT gauge
  2. Equations of motion in harmonic gauge

← **Hamiltonian**

← **EOM**

ADM formalism

- E.g., Damour-Jarawnoski-Schafer (2001) or references in Blanchet's LRR review.
- Lagrangian of non-spinning particles interacting through gravity.

$$L = \sum_A p_A \frac{dx_A}{dt} + \int d^3x(\pi^{ij} g_{ij\,,0} - N_\alpha \mathcal{H}^\alpha) - \oint d^2 s_i\, \partial_j (g_{ij} - g_{kk}\delta_{ij})$$

- N_a is lapse and shift, $\boldsymbol{\pi}^{ij}$ are conjugate momenta of the metric.
- Hamiltonian from the Lagrangian

$$H = \int d^3x N_\alpha \mathcal{H}^\alpha + \oint d^2 s_i\, \partial_j (g_{ij} - g_{kk}\delta_{ij})$$

- Then solving the constraint equation using post-Newtonian approximation assuming asymptotically Minkowskian coordinates, we obtain the reduce Hamiltonian that governs the orbital dynamics. $H_{\mathrm{reduced}} = E[h_{ij}^{\mathrm{TT}}, \pi^{ij\,\mathrm{TT}}, x_A, p_A]$

Why need to derive EOM in GR using PNA???

Is not the EOM just a geodesic equation????

Yes for a test particle. A test particle follows a geodesic of background space-time.

But for an equal/comparable mass binary, binary components follow a "geodesic", even if it is the case, of what space-time?? Is there any "background space-time"?

Why need to derive EOM in GR using PNA???

Linear order metric $g = g_{father} + g_{mother}$

Leading order geodesic equation

$$a_f = -\boldsymbol{\Gamma}[g]u[g]u[g] = -\boldsymbol{\Gamma}[g_m]u[g_m]u[g_m]$$: maybe ok?

Higher order $g = g_{father} + g_{mother} + g_{chidren} + g_{grand\text{-}chidren} + \ldots$

Higher order geodesic equation ????

$$a_f = -\boldsymbol{\Gamma}[g]u[g]u[g] = -(\boldsymbol{\Gamma}uu)[g_m+g_c+g_{gc}+\ldots]$$: this seems wrong both mathematically and theoretically.

Why need to derive EOM in GR using PNA???

Higher order geodesic equation ????

$$a_f = -\,\Gamma[g]u[g]u[g] = -\,(\Gamma uu)[g_m + g_c + g_{gc} + \ldots].$$

This is unsatisfactory.

- $g_m + g_c + g_{gc} + \ldots$ is NOT a solution of the Einstein equations!
- It is not symmetric to a_m.
- We would double count if we sum up $m_m a_m + m_f a_f$.
- If we use a Dirac delta functional to represent a point particle, then parts of $g_c + g_{gc} + \ldots$ diverge on the point particle.