Post-Newtonian approximation *and gravitational waves from compact binaries II.*

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contents:

POST-NEWTONIAN EQUATIONS OF MOTION

3.5 PN EOM (monopole terms)

1PN

$$
\begin{aligned}
m_1\frac{dv_1^i}{dt} = &-\frac{m_1m_2}{r_{12}^2}n_{12}^i\\
&+\epsilon^2\frac{m_1m_2}{r_{12}^2}n_{12}^i\left[-v_1^2-2v_2^2+4(\vec v_1\cdot\vec v_2)+\frac{3}{2}(\vec n_{12}\cdot\vec v_2)^2+\frac{5m_1}{r_{12}}+\frac{4m_2}{r_{12}}\right]\\
&+\epsilon^2\frac{m_1m_2}{r_{12}^2}V^i\left[4(\vec n_{12}\cdot\vec v_1)-3(\vec n_{12}\cdot\vec v_2)\right]
\end{aligned}
$$

2PN

$$
\begin{aligned}
&+\epsilon^4\frac{m_1m_2}{r_{12}^2}n_{12}^i\Big[-2v_2^4+4v_2^2(\vec v_1\cdot\vec v_2)-2(\vec v_1\cdot\vec v_2)^2+\frac{3}{2}v_1^2(\vec n_{12}\cdot\vec v_2)^2+\frac{9}{2}v_2^2(\vec n_{12}\cdot\vec v_2)^2\\
&\quad-6(\vec v_1\cdot\vec v_2)(\vec n_{12}\cdot\vec v_2)^2-\frac{15}{8}(\vec n_{12}\cdot\vec v_2)^4-\frac{57}{4}\frac{m_1^2}{r_{12}^2}-9\frac{m_2^2}{r_{12}^2}-\frac{69}{2}\frac{m_1m_2}{r_{12}^2}\\
&\quad+\frac{m_1}{r_{12}}\left(-\frac{15}{4}v_1^2+\frac{5}{4}v_2^2-\frac{5}{2}(\vec v_1\cdot\vec v_2)+\frac{39}{2}(\vec n_{12}\cdot\vec v_1)^2\right.\\
&\qquad\left.-39(\vec n_{12}\cdot\vec v_1)(\vec n_{12}\cdot\vec v_2)+\frac{17}{2}(\vec n_{12}\cdot\vec v_2)^2\right)\\
&\quad+\frac{m_2}{r_{12}}\left(4v_2^2-8(\vec v_1\cdot\vec v_2)+2(\vec n_{12}\cdot\vec v_1)^2-4(\vec n_{12}\cdot\vec v_1)(\vec n_{12}\cdot\vec v_2)-6(\vec n_{12}\cdot\vec v_2)^2\right)\Big]\\
&+\epsilon^4\frac{m_1m_2}{r_{12}^2}V^i\left[\frac{m_1}{r_{12}}\left(-\frac{63}{4}(\vec n_{12}\cdot\vec v_1)+\frac{55}{4}(\vec n_{12}\cdot\vec v_2)\right)+\frac{m_2}{r_{12}}\left(-2(\vec n_{12}\cdot\vec v_1)-2(\vec n_{12}\cdot\vec v_2)\right)\right.\\
&\quad+v_1^2(\vec n_{12}\cdot\vec v_2)+4v_2^2(\vec n_{12}\cdot\vec v_1)-5v_2^2(\vec n_{12}\cdot\vec v_2)-4(\vec v_1\cdot\vec v_2)(\vec n_{12}\cdot\vec V)\\
&\quad\left.-6(\vec n_{12}\cdot\vec v_1)(\vec n_{12}\cdot\vec v_2)^2+\frac{9}{2}(\vec n_{12}\cdot\vec v_2)^3\right]
\end{aligned}
$$

2.5PN

$$
+\epsilon^5\frac{4m_1^2m_2}{5r_{12}^3}\left[n_{12}^i(\vec n_{12}\cdot\vec V)\left(-6\frac{m_1}{r_{12}}+\frac{52}{3}\frac{m_2}{r_{12}}+3V^2\right)+V^i\left(2\frac{m_1}{r_{12}}-8\frac{m_2}{r_{12}}-V^2\right)\right]
$$

3PN

$$
\begin{aligned}
&+\epsilon^6\frac{m_1m_2}{r_{12}^2}n_{12}^i\Big[\frac{35}{16}(\vec n_{12}\cdot\vec v_2)^6-\frac{15}{8}(\vec n_{12}\cdot\vec v_2)^4v_1^2+\frac{15}{2}(\vec n_{12}\cdot\vec v_2)^4(\vec v_1\cdot\vec v_2)\\
&\quad+3(\vec n_{12}\cdot\vec v_2)^2(\vec v_1\cdot\vec v_2)^2-\frac{15}{2}(\vec n_{12}\cdot\vec v_2)^4v_2^2+\frac{3}{2}(\vec n_{12}\cdot\vec v_2)^2v_1^2v_2^2\\
&\quad-12(\vec n_{12}\cdot\vec v_2)^2(\vec v_1\cdot\vec v_2)v_2^2-2(\vec v_1\cdot\vec v_2)^2v_2^2+\frac{15}{2}(\vec n_{12}\cdot\vec v_2)^2v_2^4\\
&\quad+4(\vec v_1\cdot\vec v_2)v_2^4-2v_2^6\\
&\quad+\frac{m_1}{r_{12}}\left(-\frac{171}{8}(\vec n_{12}\cdot\vec v_1)^4+\frac{171}{2}(\vec n_{12}\cdot\vec v_1)^3(\vec n_{12}\cdot\vec v_2)\right.\\
&\qquad-\frac{723}{4}(\vec n_{12}\cdot\vec v_1)^2(\vec n_{12}\cdot\vec v_2)^2+\frac{383}{2}(\vec n_{12}\cdot\vec v_1)(\vec n_{12}\cdot\vec v_2)^3\\
&\qquad-\frac{455}{8}(\vec n_{12}\cdot\vec v_2)^4+\frac{229}{4}(\vec n_{12}\cdot\vec v_1)^2v_1^2-\frac{205}{2}(\vec n_{12}\cdot\vec v_1)(\vec n_{12}\cdot\vec v_2)v_1^2\\
&\qquad+\frac{191}{4}(\vec n_{12}\cdot\vec v_2)^2v_1^2-\frac{91}{8}v_1^4-\frac{229}{2}(\vec n_{12}\cdot\vec v_1)^2(\vec v_1\cdot\vec v_2)\\
&\qquad+244(\vec n_{12}\cdot\vec v_1)(\vec n_{12}\cdot\vec v_2)(\vec v_1\cdot\vec v_2)-\frac{225}{2}(\vec n_{12}\cdot\vec v_2)^2(\vec v_1\cdot\vec v_2)\\
&\qquad+\frac{91}{2}v_1^2(\vec v_1\cdot\vec v_2)-\frac{177}{4}(\vec v_1\cdot\vec v_2)^2+\frac{229}{4}(\vec n_{12}\cdot\vec v_1)^2v_2^2\\
&\qquad-\frac{283}{2}(\vec n_{12}\cdot\vec v_1)(\vec n_{12}\cdot\vec v_2)v_2^2+\frac{259}{4}(\vec n_{12}\cdot\vec v_2)^2v_2^2-\frac{91}{4}v_1^2v_2^2\\
&\qquad\left.+43(\vec v_1\cdot\vec v_2)v_2^2-\frac{81}{8}v_2^4\right)\\
&\quad+\frac{m_2}{r_{12}}\left(-6(\vec n_{12}\cdot\vec v_1)^2(\vec n_{12}\cdot\vec v_2)^2+12(\vec n_{12}\cdot\vec v_1)(\vec n_{12}\cdot\vec v_2)^3\right.\\
&\qquad+6(\vec n_{12}\cdot\vec v_2)^4+4(\vec n_{12}\cdot\vec v_1)(\vec n_{12}\cdot\vec v_2)(\vec v_1\cdot\vec v_2)\\
&\qquad+12(\vec n_{12}\cdot\vec v_2)^2(\vec v_1\cdot\vec v_2)+4(\vec v_1\cdot\vec v_2)^2\\
&\qquad\left.-4(\vec n_{12}\cdot\vec v_1)(\vec n_{12}\cdot\vec v_2)v_2^2-12(\vec n_{12}\cdot\vec v_2)^2v_2^2-8(\vec v_1\cdot\vec v_2)v_2^2+4v_2^4\right)\\
&\quad+\frac{m_2^2}{r_{12}^2}\left(-(\vec n_{12}\cdot\vec v_1)^2+2(\vec n_{12}\cdot\vec v_1)(\vec n_{12}\cdot\vec v_2)+\frac{43}{2}(\vec n_{12}\cdot\vec v_2)^2\right.\\
&\qquad\left.+18(\vec v_1\cdot\vec v_2)-9v_2^2\right)\\
&\quad+\frac{m_1m_2}{r_{12}^2}\left(\frac{415}{8}(\vec n_{12}\cdot\vec v_1)^2-\frac{375}{4}(\vec n_{12}\cdot\vec v_1)(\vec n_{12}\cdot\vec v_2)+\frac{1113}{8}(\vec n_{12}\cdot\vec v_2)^2\right.\\
&\qquad\left.-\frac{615\pi^2}{64}(\vec n_{12}\cdot\vec V)^2+18v_1^2+\frac{123\pi^2}{64}V^2+33(\vec v_1\cdot\vec v_2)-\frac{33}{2}v_2^2\right)\\
&\quad+\frac{m_1^2}{r_{12}^2}\left(-\frac{2069}{8}(\vec n_{12}\cdot\vec v_1)^2+543(\vec n_{12}\cdot\vec v_1)(\vec n_{12}\cdot\vec v_2)-\frac{939}{4}(\vec n_{12}\cdot\vec v_2)^2\right.\\
&\qquad\left.+\frac{471}{8}v_1^2-\frac{357}{4}(\vec v_1\cdot\vec v_2)+\frac{357}{8}v_2^2\right)\\
&\quad+\frac{16m_2^3}{r_{12}^3}+\frac{m_1^2m_2}{r_{12}^3}\left(\frac{547}{3}-\frac{41\pi^2}{16}\right)-\frac{13m_1^3}{12r_{12}^3}+\frac{m_1m_2^2}{r_{12}^3}\left(\frac{545}{3}-\frac{41\pi^2}{16}\right)\Big]\\
&+\epsilon^6\frac{m_1m_2}{r_{12}^2}V^i\Big[\frac{15}{2}(\vec n_{12}\cdot\vec v_1)(\vec n_{12}\cdot\vec v_2)^4-\frac{45}{8}(\vec n_{12}\cdot\vec v_2)^5-\frac{3}{2}(\vec n_{12}\cdot\vec v_2)^3v_1^2\\
&\quad+6(\vec n_{12}\cdot\vec v_1)(\vec n_{12}\cdot\vec v_2)^2(\vec v_1\cdot\vec v_2)-6(\vec n_{12}\cdot\vec v_2)^3(\vec v_1\cdot\vec v_2)\\
&\quad-2(\vec n_{12}\cdot\vec v_2)(\vec v_1\cdot\vec v_2)^2-12(\vec n_{12}\cdot\vec v_1)(\vec n_{12}\cdot\vec v_2)^2v_2^2+12(\vec n_{12}\cdot\vec v_2)^3v_2^2\\
&\quad+(\vec n_{12}\cdot\vec v_2)v_1^2v_2^2-4(\vec n_{12}\cdot\vec v_1)(\vec v_1\cdot\vec v_2)v_2^2+8(\vec n_{12}\cdot\vec v_2)(\vec v_1\cdot\vec v_2)v_2^2\\
&\quad+4(\vec n_{12}\cdot\vec v_1)v_2^4-7(\vec n_{12}\cdot\vec v_2)v_2^4\\
&\quad+\frac{m_2}{r_{12}}\left(-2(\vec n_{12}\cdot\vec v_1)^2(\vec n_{12}\cdot\vec v_2)+8(\vec n_{12}\cdot\vec v_1)(\vec n_{12}\cdot\vec v_2)^2+2(\vec n_{12}\cdot\vec v_2)^3\right.\\
&\qquad+2(\vec n_{12}\cdot\vec v_1)(\vec v_1\cdot\vec v_2)+4(\vec n_{12}\cdot\vec v_2)(\vec v_1\cdot\vec v_2)\\
&\qquad\left.-2(\vec n_{12}\cdot\vec v_1)v_2^2-4(\vec n_{12}\cdot\vec v_2)v_2^2\right)\\
&\quad+\frac{m_1}{r_{12}}\left(-\frac{243}{4}(\vec n_{12}\cdot\vec v_1)^3+\frac{565}{4}(\vec n_{12}\cdot\vec v_1)^2(\vec n_{12}\cdot\vec v_2)\right.\\
&\qquad-\frac{269}{4}(\vec n_{12}\cdot\vec v_1)(\vec n_{12}\cdot\vec v_2)^2-\frac{95}{12}(\vec n_{12}\cdot\vec v_2)^3+\frac{207}{8}(\vec n_{12}\cdot\vec v_1)v_1^2\\
&\qquad-\frac{137}{8}(\vec n_{12}\cdot\vec v_2)v_1^2-36(\vec n_{12}\cdot\vec v_1)(\vec v_1\cdot\vec v_2)+\frac{27}{4}(\vec n_{12}\cdot\vec v_2)(\vec v_1\cdot\vec v_2)\\
&\qquad\left.+\frac{81}{8}(\vec n_{12}\cdot\vec v_1)v_2^2+\frac{83}{8}(\vec n_{12}\cdot\vec v_2)v_2^2\right)\\
&\quad+\frac{m_2^2}{r_{12}^2}\left(4(\vec n_{12}\cdot\vec v_1)+5(\vec n_{12}\cdot\vec v_2)\right)\\
&\quad+\frac{m_1m_2}{r_{12}^2}\left(-\frac{307}{8}(\vec n_{12}\cdot\vec v_1)+\frac{479}{8}(\vec n_{12}\cdot\vec v_2)+\frac{123\pi^2}{32}(\vec n_{12}\cdot\vec V)\right)\\
&\quad+\frac{m_1^2}{r_{12}^2}\left(\frac{311}{4}(\vec n_{12}\cdot\vec v_1)-\frac{357}{4}(\vec n_{12}\cdot\vec v_2)\right)\Big]
\end{aligned}
$$

3.5PN

$$
\begin{aligned}
&+\frac{m_1^4m_2}{r_{12}^5}\left[n_{12}^i\left\{\frac{3992}{105}(\vec n_{12}\cdot\vec v_1)-\frac{4328}{105}(\vec n_{12}\cdot\vec v_2)\right\}-\frac{184}{21}V^i\right]+\frac{m_1^3m_2^2}{r_{12}^5}\left[\frac{6224}{105}V^i\right.\\
&\left.+n_{12}^i\left\{\frac{2872}{21}(\vec n_{12}\cdot\vec v_2)-\frac{13576}{105}(\vec n_{12}\cdot\vec v_1)\right\}\right]+\frac{m_1^3m_2}{r_{12}^4}V^i\left[-\frac{132}{35}v_1{}^2-\frac{48}{35}v_2{}^2+\frac{52}{15}(\vec n_{12}\cdot\vec v_1)^2\right.\\
&\left.+\frac{152}{35}(\vec v_1\cdot\vec v_2)-\frac{56}{15}(\vec n_{12}\cdot\vec v_1)(\vec n_{12}\cdot\vec v_2)-\frac{44}{15}(\vec n_{12}\cdot\vec v_2)^2\right]+\frac{m_1^3m_2}{r_{12}^4}n_{12}^i\left[-\frac{4888}{105}(\vec n_{12}\cdot\vec v_1)v_1{}^2\right.\\
&+\frac{5056}{105}(\vec n_{12}\cdot\vec v_2)v_1{}^2-\frac{1028}{21}v_2{}^2(\vec n_{12}\cdot\vec v_1)+48(\vec n_{12}\cdot\vec v_1)^3+\frac{5812}{105}v_2{}^2(\vec n_{12}\cdot\vec v_2)+\frac{2056}{21}(\vec n_{12}\cdot\vec v_1)(\vec v_1\cdot\vec v_2)\\
&\left.-\frac{2224}{21}(\vec n_{12}\cdot\vec v_2)(\vec v_1\cdot\vec v_2)-\frac{696}{5}(\vec n_{12}\cdot\vec v_1)^2(\vec n_{12}\cdot\vec v_2)+\frac{744}{5}(\vec n_{12}\cdot\vec v_1)(\vec n_{12}\cdot\vec v_2)^2-\frac{288}{5}(\vec n_{12}\cdot\vec v_2)^3\right]\\
&+\frac{m_1^2m_2^3}{r_{12}^5}\left[\frac{6388}{105}V^i-\frac{3172}{21}(\vec n_{12}\cdot\vec V)n_{12}^i\right]+\frac{m_1^2m_2}{r_{12}^3}V^i\left[\frac{334}{35}v_1{}^4+\frac{654}{35}v_2{}^2v_1{}^2-\frac{1336}{35}(\vec v_1\cdot\vec v_2)v_1{}^2+\frac{292}{35}v_2{}^4\right.\\
&-\frac{348}{5}(\vec n_{12}\cdot\vec v_1)^2V^2+60(\vec n_{12}\cdot\vec V)^4-\frac{1252}{35}v_2{}^2(\vec v_1\cdot\vec v_2)+\frac{684}{5}(\vec n_{12}\cdot\vec v_1)(\vec n_{12}\cdot\vec v_2)V^2+\frac{1308}{35}(\vec v_1\cdot\vec v_2)^2\\
&\left.-66(\vec n_{12}\cdot\vec v_2)^2V^2\right]+\frac{m_1^2m_2}{r_{12}^3}n_{12}^i\left[-\frac{246}{35}(\vec n_{12}\cdot\vec V)v_1{}^4-\frac{534}{35}v_2{}^2(\vec n_{12}\cdot\vec v_1)v_1{}^2+\frac{90}{7}v_2{}^2(\vec n_{12}\cdot\vec v_2)v_1{}^2\right.\\
&+\frac{1068}{35}(\vec n_{12}\cdot\vec v_1)(\vec v_1\cdot\vec v_2)v_1{}^2-\frac{984}{35}(\vec n_{12}\cdot\vec v_2)(\vec v_1\cdot\vec v_2)v_1{}^2-\frac{204}{35}v_2{}^4(\vec n_{12}\cdot\vec v_1)+60(\vec n_{12}\cdot\vec v_1)^3V^2\\
&-56(\vec n_{12}\cdot\vec V)^5+\frac{24}{7}v_2{}^4(\vec n_{12}\cdot\vec v_2)+\frac{984}{35}v_2{}^2(\vec n_{12}\cdot\vec v_1)(\vec v_1\cdot\vec v_2)-\frac{732}{35}v_2{}^2(\vec n_{12}\cdot\vec v_2)(\vec v_1\cdot\vec v_2)\\
&-180(\vec n_{12}\cdot\vec v_1)^2(\vec n_{12}\cdot\vec v_2)V^2-\frac{1068}{35}(\vec n_{12}\cdot\vec v_1)(\vec v_1\cdot\vec v_2)^2+174(\vec n_{12}\cdot\vec v_1)(\vec n_{12}\cdot\vec v_2)^2V^2\\
&\left.+\frac{180}{7}(\vec n_{12}\cdot\vec v_2)(\vec v_1\cdot\vec v_2)^2-54V^2(\vec n_{12}\cdot\vec v_2)^3\right]+\frac{m_1^2m_2{}^2}{r_{12}^4}V^i\left[-\frac{152}{21}v_1{}^2-\frac{1768}{105}v_2{}^2+\frac{454}{15}(\vec n_{12}\cdot\vec v_1)^2\right.\\
&\left.+\frac{2864}{105}(\vec v_1\cdot\vec v_2)-\frac{372}{5}(\vec n_{12}\cdot\vec v_1)(\vec n_{12}\cdot\vec v_2)+\frac{854}{15}(\vec n_{12}\cdot\vec v_2)^2\right]+\frac{m_1^2m_2{}^2}{r_{12}^4}n_{12}^i\left[\frac{1432}{35}(\vec n_{12}\cdot\vec v_1)v_2{}^2\right.\\
&-\frac{5752}{105}(\vec n_{12}\cdot\vec v_2)v_2{}^2-\frac{582}{5}(\vec n_{12}\cdot\vec v_1)^3+\frac{3568}{105}(\vec n_{12}\cdot\vec V)v_1^2-\frac{2864}{35}(\vec n_{12}\cdot\vec v_1)(\vec v_1\cdot\vec v_2)\\
&+\frac{10048}{105}(\vec n_{12}\cdot\vec v_2)(\vec v_1\cdot\vec v_2)+\frac{1746}{5}(\vec n_{12}\cdot\vec v_1)^2(\vec n_{12}\cdot\vec v_2)-\frac{1954}{5}(\vec n_{12}\cdot\vec v_1)(\vec n_{12}\cdot\vec v_2)^2+158(\vec n_{12}\cdot\vec v_2)^{\cdot}
\end{aligned}
$$

Post-Newtonian equations of motion for relativistic compact binaries

- Plan: Concentrate on (my contributions to) 3.5 PN EOM.

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For other approaches, see e.g. L. Blanchet, Living Review in Relativity 9, 4 (2016), & Maggiore's text book.

Key ideas in our formalism

1. Hyperbolic formulation of Post-Newtonian approximation (PNA)

   - Anderson & Decanio (1975)

2. Point particle limit

   - Strong field point particle limit (Futamase,1987)

3. Surface integral approach

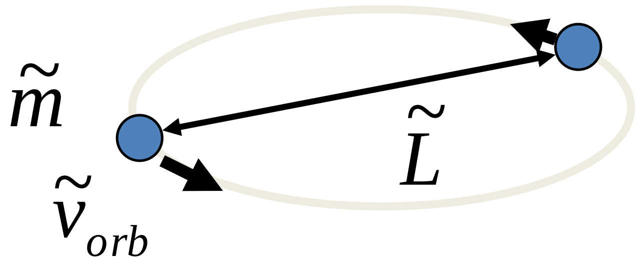
   - Similar to Einstein, Infeld & Hoffmann (1938).

Key ideas 1. PNA

- Post-Newtonian approximation.

Newtonian gravitational bound system : Balance between centrifugal force and gravitational force

$$\frac{G\tilde{m}}{c^2\tilde{L}} \simeq \left(\frac{\tilde{v}_{orb}}{c}\right)^2$$



Introduce scaled mass m and velocity v, PN Expansion parameter ε , Newtonian dynamical time $\boxed{\tau}$ [Nothing to do with the proper time!!].

$$\tilde{v}_{\rm orb} \equiv \frac{dx^i}{dt} \equiv \epsilon\frac{dx^i}{d\tau} \equiv \epsilon v^i, \quad \tilde{m} \equiv \epsilon^2 m$$

Key ideas 1. PNA (cont'd)

1. Expanding metric (and stress energy tensor) in ε formally.

Use lower order EOM if necessary

$$g(x;\epsilon) = \eta + \epsilon\partial_\epsilon g(x;0) + \epsilon^2\partial_\epsilon^2 g(x;0)/2! \cdots,$$

Iteration

2. Expanding Einstein Equations in ε.

Solve those for g_n's up to required order in ε as functional of m, v, …

$$G[g(x,\epsilon)] = T(x;\epsilon^2 m, \epsilon v, \cdots)$$

3. Stress energy conservation law gives EOM.

$$\nabla_{[g(x;\epsilon)]} T(x;\epsilon^2 m, \epsilon v, \cdots) = 0$$

$$m_1 \frac{dv_1^i}{d\tau} = -\frac{m_1 m_2}{r_{12}^3} r_{12}^i + \epsilon^2 F_{1\mathrm{PN}} + \epsilon^4 F_{2\mathrm{PN}} + \epsilon^5 F_{2.5\mathrm{PN}} + \epsilon^6 F_{3\mathrm{PN}} + \epsilon^7 F_{3.5\mathrm{PN}} + \cdots$$

Key ideas 2. Point particle limit

Why point particle limit?

1. **To make equations of motion more tractable (reduce number of degrees of freedom)**

2. **Gravitational wave data analysis may not need higher order multipoles other than spin (and quadrupole). Smaller the number of parameters (mass, spins, …) to be searched for is, easier the data analysis and lesser the computational burden become.**

One possible approach: Dirac delta

- One can use Dirac delta to achieve point particle limit.
- Have to deal with divergent integrals.

$$g(x) \sim 1 - \frac{2Gm}{|\vec{x} - \vec{z}(t)|} + \ldots$$

$$\int T^{\mu\nu} d^3x = \int \frac{m v^\mu v^\nu}{\sqrt{-g g_{\rho\sigma} v^\rho v^\sigma}} \delta_D(\vec{x} - \vec{z}(t)) d^3x$$

- Up to the 2.5 PN order (and 3.5 PN order), it was sufficient to use the Hadamard Partie Finie regularization.
- At the 3PN and 4PN order, we need the dimensional regularization.

Key ideas 2. Point particle limit (cont'd)

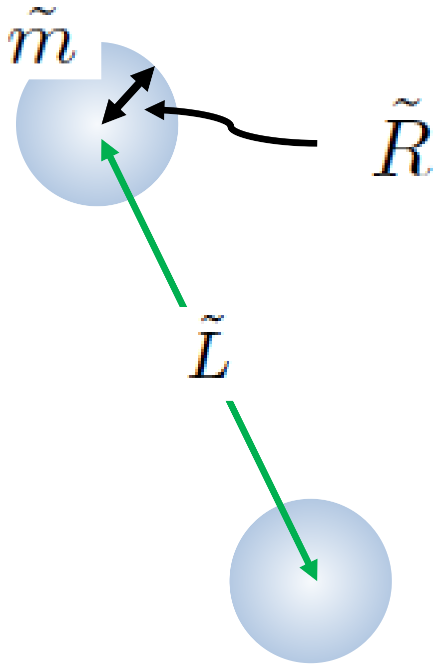
Strong field point particle limit:

- **"Regular" point particle limit.**
- **Can make a star have strong internal self-gravity (while keeping inter-body gravity weak).**
- **Nicely fit into post-Newtonian approximation.**

1. We would like to make a star have strong internal gravity $\frac{\tilde{m}}{\tilde{R}} = O(1)$

2. while keeping inter-star gravity weak and PNA valid. $\frac{\tilde{m}}{\tilde{L}} = O(\epsilon^2)$

3. As a consequence, we have a point particle in the ε-zero limit. $\frac{\tilde{R}}{\tilde{L}} = O(\epsilon^2)$



Scaling law for radius of star: (Strong field point particle limit)

$$\tilde{R}_A \equiv \epsilon^2 \bar{R}_A$$

Key ideas 3. Surface integral approach

- (Newtonian) Force by Volume integral

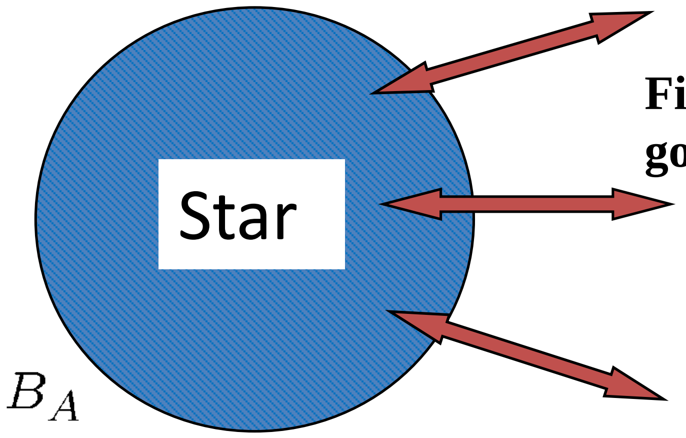
$$F_1^i = -\int_{\mathrm{B}_1} d^3x \rho \frac{\partial \phi}{\partial x^i}$$

Need ρ and φ inside the star.

- By surface integral (using Poisson eq.)

$$F_1^i = -\oint_{\partial B_1} dS_j t^{ij},$$
$$t^{ij} \equiv \frac{1}{4\pi}\left(\frac{\partial \phi}{\partial x^i}\frac{\partial \phi}{\partial x^j} - \frac{\delta^{ij}}{2}\frac{\partial \phi}{\partial x^k}\frac{\partial \phi}{\partial x^k}\right).$$

Need φ close but outside the star.



Field Momentum flux going through B_A

B_A (or star A) shrinks

Gravitational Force on the star A

Ways to EOM

(1)Volume integral Approach: (Pati & Will)

Assume the properties of the density.

$$F_1^i = \int_{B_1} d^3x \rho(x) \frac{\partial \phi}{\partial x^i}.$$

Explicit demonstration of irrelevance of the internal structure.

(2) Regularized geodesics or, regularized action (Blanchet & Faye)

$$[u^\nu u^\mu{}_{;\nu}]^{reg} = 0$$

Physically interesting implications.

(3)Surface Integral Approach: (Einstein, Infeld & Hoffmann, YI, Futamase & Asada)

$$F_1^i = -\oint_{\partial B_1} dS_j t^{ij},$$

$$t^{ij} \equiv \frac{1}{4\pi}\left(\frac{\partial \phi}{\partial x^i}\frac{\partial \phi}{\partial x^j} - \frac{\delta^{ij}}{2}\frac{\partial \phi}{\partial x^k}\frac{\partial \phi}{\partial x^k}\right).$$

Avoid the internal problem up to the order where φ depends on it.

(4) Effective Field Theoretical approach

Newtonian computations.

mass $m_A = \int_{B_A} d^3x\rho$

dipole $D^i_A \equiv \int_{B_A} d^3x\rho(x^i - z^i_A)$

momentum $P^i_A \equiv \int_{B_A} d^3x\rho v^i$

Velocity Momentum relation

$$\frac{dD^i_A}{dt} = P^i_A - m_A v^i_A = 0$$

From Field eq. $\phi = \sum_{A=1,2}\left[\frac{m_A}{|\vec{x}-\vec{z}_A|} + \frac{1}{2} I^{ij}_A \frac{\partial^2}{\partial x^i x^j}\left(\frac{1}{|\vec{x}-\vec{z}_A|}\right) + \cdots\right].$

$$F^i_1 \;=\; -\oint_{\partial B_1} dS_j t^{ij},$$

$$t^{ij} \;\equiv\; \frac{1}{4\pi}\left(\frac{\partial\phi}{\partial x^i}\frac{\partial\phi}{\partial x^j} - \frac{\delta^{ij}}{2}\frac{\partial\phi}{\partial x^k}\frac{\partial\phi}{\partial x^k}\right).$$

$$m_1\frac{d^2 z^i_1}{d\tau^2} \;=\; \sum_{p=0}\sum_{q=0}\frac{(-1)^{p+1}(2p+2q+1)!!}{p!q!}\frac{I_1^{<M_p>} I_2^{<N_q>}}{r_{12}^{p+q+2}} N^{<iM_pN_q>}$$

Field equations

1. Gauge choice
2. Relxed Einstein Equations (REE)
3. How to solve REE.
   - Boundary conditions
   - How to deal with PNA break down
   - Field around stars: operational multipole moments.
   - Super(-duper-tuper-…)- potentials.
   - PNA iteration

Field Equation

•**Deviation field h.** $h^{\mu\nu} \equiv \eta^{\mu\nu} - \sqrt{-g}g^{\mu\nu}$

•**Harmonic gauge** ${h^{\mu\nu}}_{,\nu} = 0,$

•**Relaxed Einstein Equations (REE)**
Anderson & Decanio (1975).

$$\Box h^{\mu\nu} = -16\pi\Lambda^{\mu\nu}$$

flat wave operator

REE source terms

$$\Lambda^{\mu\nu} \equiv \Theta^{\mu\nu} + {\chi^{\mu\nu\alpha\beta}}_{,\alpha\beta},$$

$$\Theta^{\mu\nu} \equiv (-g)(T^{\mu\nu} + t^{\mu\nu}_{LL}),$$

Stress energy tensor of matter plus gravity

$$\chi^{\mu\nu\alpha\beta} \equiv \frac{1}{16\pi}(h^{\alpha\nu}h^{\beta\mu} - h^{\alpha\beta}h^{\mu\nu}).$$

Wave operator residual.

Conservation laws $${\Lambda^{\mu\nu}}_{,\nu} = 0, {\Theta^{\mu\nu}}_{,\nu} = 0, {\chi^{\mu\nu\alpha\beta}}_{,\alpha\beta\nu} = 0.$$

•**Formal solution to REE.**

$$h^{\mu\nu}(\tau, x^i) = 4\int_{C(\tau,x^k)} d^3y \frac{\Lambda^{\mu\nu}(\tau - \epsilon|\vec{x} - \vec{y}|, y^k; \epsilon)}{|\vec{x} - \vec{y}|} + h^{\mu\nu}_H(\tau, x^i),$$

flat light cone **Homogeneous term**

Boundary condition:

•**Homogeneous solution:**

$$h_H^{\mu\nu}(\tau, x^i) \quad = \quad \oint_{\partial C(\tau, x^i)} \frac{d\Omega_y}{4\pi} \left[\frac{\partial}{\partial \rho}(\rho h^{\mu\nu}(\tau', y^i)) + \frac{\partial}{\partial \tau'}(\rho h^{\mu\nu}(\tau', y^i)) \right] \Bigg|_{\tau'=0, \rho=|\vec{x}-\vec{y}|=\tau} .$$

•**No incoming radiation condition at Minkowskian past null infinity.**

$$\lim_{\substack{\tau=r,\\ r\to\infty}} \left[\frac{\partial}{\partial r}(r h^{\mu\nu}) + \frac{\partial}{\partial \tau}(r h^{\mu\nu}) \right] = 0. \quad \textbf{or} \quad h_H^{\mu\nu} = 0.$$

Other possibilities:

• **Use "radiative coordinates" to incorporate system monopole effect on null characteristic (MPM of Blanchet, Damour, Iyer et al.).**

-- No difference in EOM up to 3.5 PN order inclusively.

• **Use initial value formalism rather than going to fictitious past null (BigBang).**

-- Assume binary is immersed in (environmental/cosmological) stochastic GWs h^{ij} (not h^{tt}, h^{ti}). (Statistical initial condition by Schutz 1980.)

-- Not deeply investigated.

Let's solve the Relaxed Einstein Equations iteratively.

$$h^{\mu\nu}(\tau, x^i) = 4 \int_{C(\tau, x^k)} d^3y \frac{\Lambda^{\mu\nu}(\tau - \epsilon|\vec{x} - \vec{y}|, y^k; \epsilon)}{|\vec{x} - \vec{y}|} + h_H^{\mu\nu}(\tau, x^i),$$

Outside of the material source, the integrand consists of at most O(h^2) (or O(G^2)).

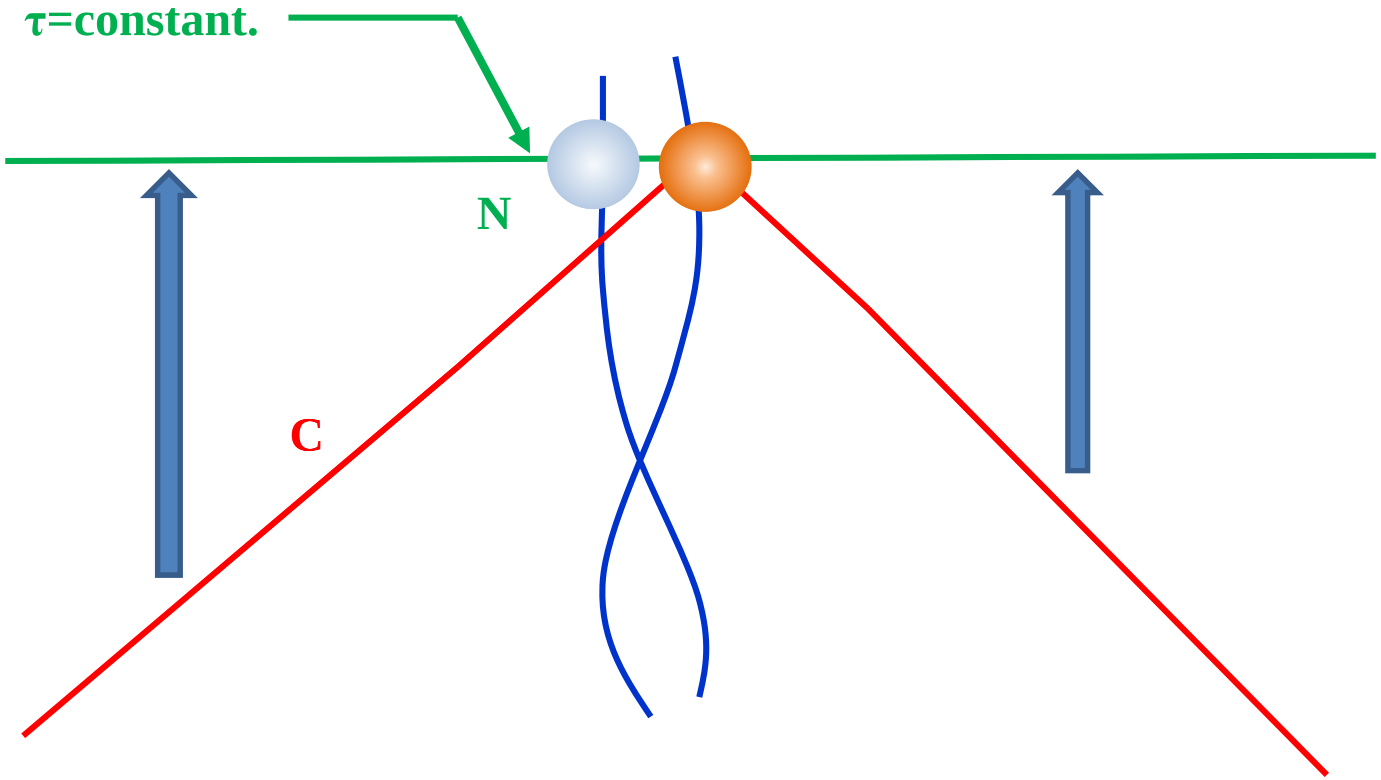
$$\Lambda \sim (\partial h)^2 + h(\partial h)^3 + \cdots$$

We can solve REE iteratively.

PNA break-down, Far zone field, & WWP-DIRE

Divergent integrals in formal slow motion expansion series

$$h(\tau, x^i) \sim \int_{C} d^3y \frac{f(\tau - \epsilon|\vec{x} - \vec{y}|, \vec{y})}{|\vec{x} - \vec{y}|} \sim \sum_{n}^{\infty} \frac{(-1)^n}{n!} \epsilon^n \int_{N} d^3y |\vec{x} - \vec{y}|^{n-1} \frac{d^n}{dt^n} f(t, \vec{y}).$$



Forward in time the integrand from C to N using slow motion expansion (Taylor expansion), and evaluate the integral on N.

PNA break-down, Far zone field, & WWP-DIRE

Divergent integrals in formal slow motion expansion series

$$h(\tau, x^i) \sim \int d^3y \frac{f(\tau - \epsilon|\vec{x} - \vec{y}|, \vec{y})}{|\vec{x} - \vec{y}|} \sim \sum_n^\infty \frac{(-1)^n}{n!} \epsilon^n \int d^3y |\vec{x} - \vec{y}|^{n-1} \frac{d^n}{dt^n} f(t, \vec{y}).$$

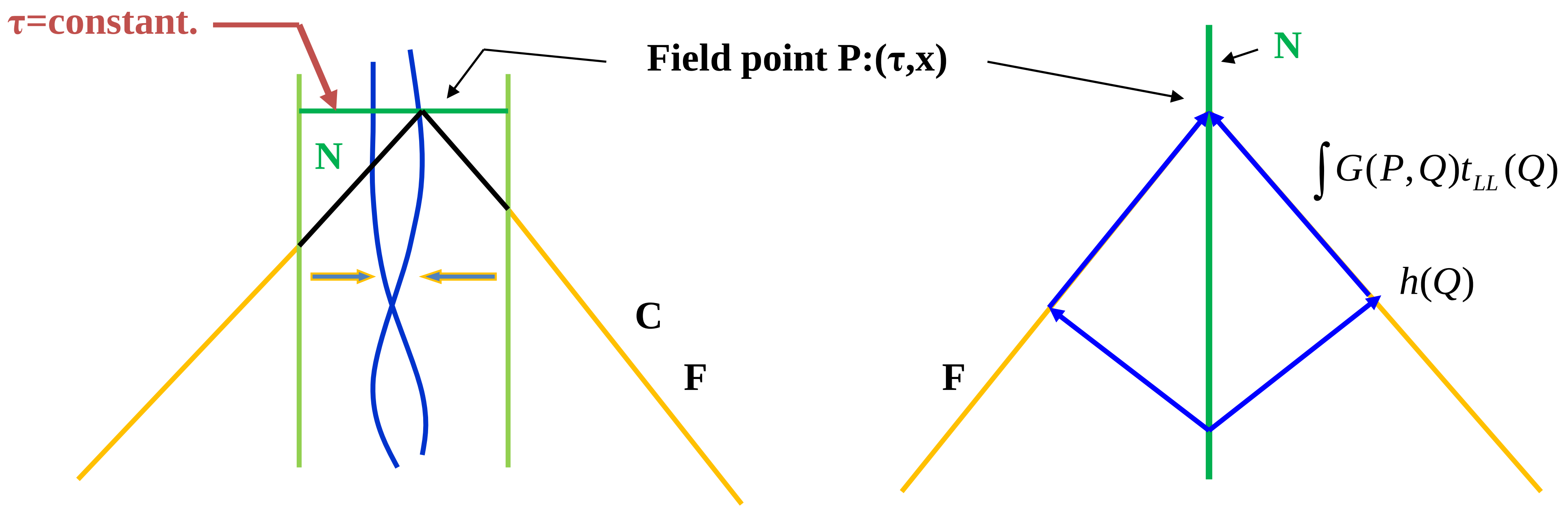
- Multipolar-Post-Minkowskian formalism (MPM)
  – Blanchet, Damour, Iyer et al. (e.g. Blanchet 2016 review)
  – PMA in radiative coordinates for far zone
  – PNA for near zone
  – Matching between two.
- Direct Integration of Relaxed Einstein Equations (DIRE)
  – Will & Wiseman (1996)
  – same coordinates in far and near zone (harmonic).

Will-Wiseman-Pati's DIRE

$$h^{\mu\nu}_{N(C)} = h^{\mu\nu}_{N(N)} + h^{\mu\nu}_{N(F)} + h^{\mu\nu}_{\rm H}\,,$$

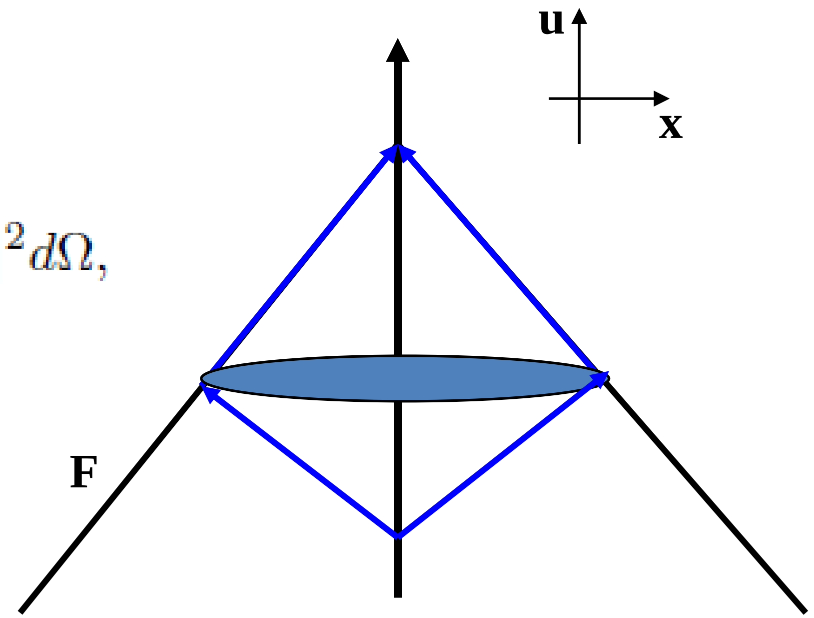
$$h^{\mu\nu}_{N(N)} = 4\int_{N=\{y:|y|\le\mathcal{R}/\epsilon\}} d^3y\,\frac{\Lambda^{\mu\nu}(\tau-\epsilon|\vec{x}-\vec{y}|,y^k;\epsilon)}{|\vec{x}-\vec{y}|}\,;$$

$$h^{\mu\nu}_{N(F)} = 4\int_{F=\{y:|y|>\mathcal{R}/\epsilon\}} d^3y\,\frac{\Lambda^{\mu\nu}(\tau-\epsilon|\vec{x}-\vec{y}|,y^k;\epsilon)}{|\vec{x}-\vec{y}|}\,,$$



Will-Wiseman-Pati's DIRE cont'd

$$
\begin{aligned}
h^{\mu\nu}_{F(F)}(t,\vec{x}) &= 4\int_F d^3x' \frac{\Lambda^{\mu\nu}(t-|\vec{x}-\vec{x}'|,\vec{x}')}{|\vec{x}-\vec{x}'|} \\
&= 4\int_{-\infty}^{u} du' \oint_F \frac{\Lambda^{\mu\nu}(u'+r',\vec{x}')}{t-u'-\vec{n}'\cdot\vec{x}}[r'(u',\Omega')]^2 d\Omega,
\end{aligned}
$$



STF expansion

$$\Lambda^{\mu\nu} \sim f_{B,L} r^{-B} n^{<L>}$$

General formula for far zone contribution to near zone field

$$
\begin{aligned}
h^{\mu\nu}_{N(F)}(t,x^i) &= \sum_{\substack{B\neq 2\\ l}} \left(\frac{2}{r}\right)^{B-2} n^{<L>} \sum_{q=0} \mathcal{E}_{B,L}(z)^q r^q \frac{d^q f_{B,L}(t)}{dt^q} \\
&+ \frac{n^{<L>}}{r}\int_0^\infty f_{2,L}(u-s) \mathrm{Q}_l\left(1+\frac{s}{r}\right) + n^{<L>}\sum_{q=0}\mathcal{E}^q_{2,L}(z) r^q \frac{d^q f_{2,L}(t)}{dt^q},
\end{aligned}
$$

PNA

appear at 4 PN EOM as PN tail.

Near zone field and slow motion expansion

- **Slow motion expansion**

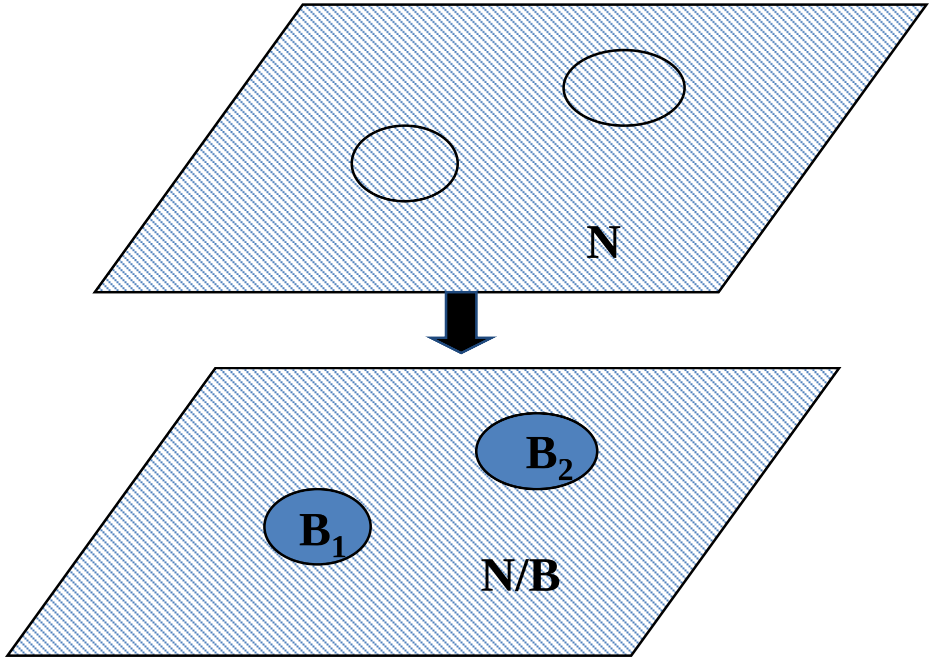
$$h^{\mu\nu} = 4\sum_{n=0} \frac{(-\epsilon)^n}{n!} \left(\frac{\partial}{\partial\tau}\right)^n \int_N d^3y |\vec{x} - \vec{y}|^{n-1} \Lambda_N^{\mu\nu}(\tau, y^k; \epsilon).$$

- **Then split it into Body zone contribution + N/B contribution**

$$h = h_B + h_{N/B},$$

$$h_B = \epsilon^6 \sum_{A=1,2} \int_{B_A} d^3\alpha_A \frac{f(\tau, \vec{z}_A + \epsilon^2\vec{\alpha}_A)}{|\vec{r}_A - \epsilon^2\vec{\alpha}_A|^{1-n}},$$

$$h_{N/B} = \int_{N/B} d^3y \frac{f(\tau, \vec{y})}{|\vec{x} - \vec{y}|^{1-n}},$$



Body zone field and multipole expansion

Body zone contribution : Multipole expansion

$$h^{\tau\tau}_{Bn=0} = 4\epsilon^4 \sum_{A=1,2} \left(\frac{P^\tau_A}{r_A} + \epsilon^2 \frac{D^k_A r^k_A}{r^3_A} + \epsilon^4 \frac{3I^{<kl>}_A r^k_A r^l_A}{2r^5_A} + \epsilon^6 \frac{5I^{<klm>}_A r^k_A r^l_A r^m_A}{2r^7_A} \right)$$

$$h^{\tau i}_{Bn=0} = 4\epsilon^4 \sum_{A=1,2} \left(\frac{P^i_A}{r_A} + \epsilon^2 \frac{J^{ki}_A r^k_A}{r^3_A} + \epsilon^4 \frac{3J^{kli}_A r^k_A r^l_A}{2r^5_A} \right)$$

$$h^{ij}_{Bn=0} = 4\epsilon^2 \sum_{A=1,2} \left(\frac{Z^{ij}_A}{r_A} + \epsilon^2 \frac{Z^{kij}_A r^k_A}{r^3_A} + \epsilon^4 \frac{3Z^{<kl>ij}_A r^k_A r^l_A}{2r^5_A} + \epsilon^6 \frac{5Z^{<klm>ij}_A r^k_A r^l_A r^m_A}{2r^7_A} \right)$$

Operational multipoles

$$I^{K_l}_A \equiv \epsilon^2 \int_{B_A} d^3\alpha_A \Lambda^{\tau\tau}_N \alpha^{K_l}_A,$$

$$J^{K_l i}_A \equiv \epsilon^4 \int_{B_A} d^3\alpha_A \Lambda^{\tau i}_N \alpha^{K_l}_A, \longleftarrow$$

$$Z^{K_l ij}_A \equiv \epsilon^8 \int_{B_A} d^3\alpha_A \Lambda^{ij}_N \alpha^{K_l}_A,$$

Integrands include gravitational stress energy tensor →

Self-gravitating star.

$$P^\tau_A \equiv I^{K_0}_A$$

$$D^{k_1}_A \equiv I^{K_1}_A$$

$$P^{k_1}_A \equiv J^{K_1}_A$$

moments of stars

Conservation law gives the velocity-momentum relation

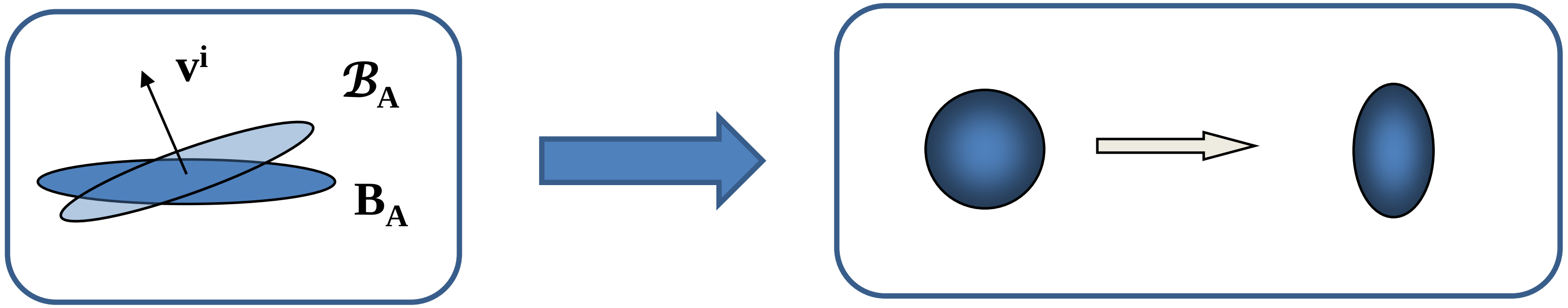
$$P^i_A = P^\tau v^i_A + Q^i_A + \frac{dD^i_A}{d\tau}$$

Relativistic correction due to the gravitational energy.

Define the moments of the star A in its Fermi normal coordinates. Namely, spin. Choose some particular spin condition.

$$\mathcal{S}_{A\mu} u^\mu_A = 0 \quad \text{, or equivalently,} \quad \mathcal{D}^\mu_A = -\mathcal{M}^{\mu\nu}_A u_{A\mu} = 0,$$

We want a "spherical" object in its rest frame when neglecting higher order multipoles.

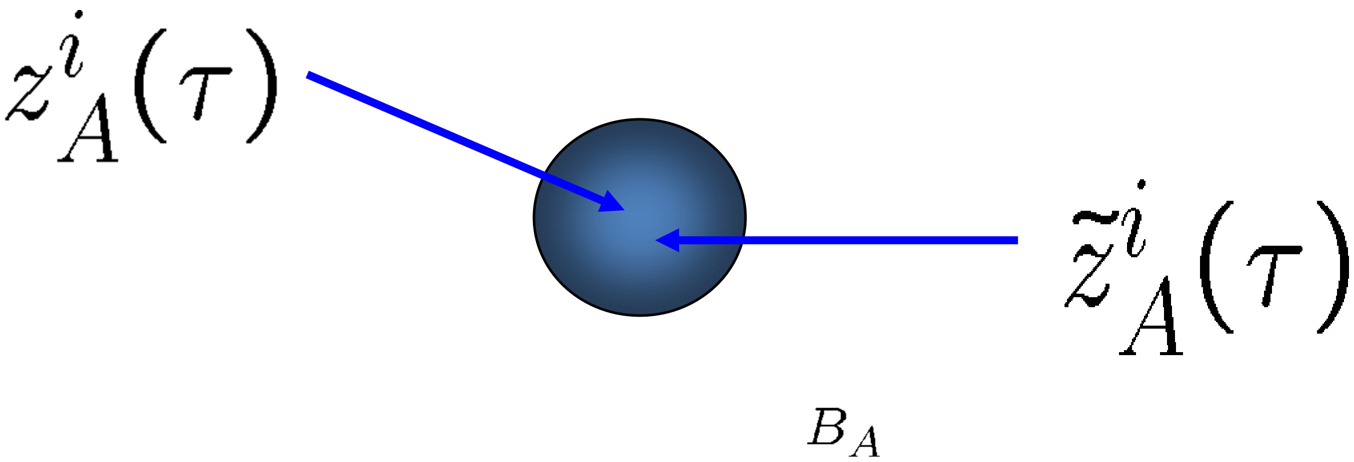


Momentum Velocity relation and a representative pint of the star

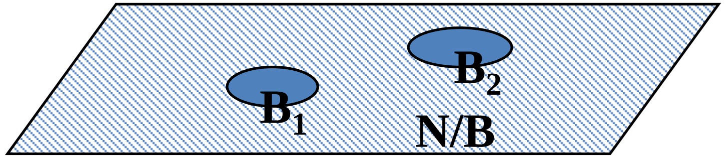
$$P^i_{\ominus A} = P^\tau_{\ominus A} v^i_A + Q^i_{\ominus A} + \epsilon^2 \frac{dD^i_{\ominus A}}{d\tau},$$

P is not proportional to v.

Need to care for which point in the star is representative.
→Specify the dipole moment freely and determine which point inside the star represents the star in the point particle limit.



N/B field and super-potentials



$$\int_{N/B} d^3y \frac{f(\vec{y})}{|\vec{x}-\vec{y}|} = -4\pi g(\vec{x}) + \oint_{\partial(N/B)} dS_k \left[\frac{1}{|\vec{x}-\vec{y}|}\frac{\partial g(\vec{y})}{\partial y^k} - g(\vec{y})\frac{\partial}{\partial y^k}\left(\frac{1}{|\vec{x}-\vec{y}|}\right)\right].$$

$$\Delta g(\vec{x}) = f(\vec{x})$$

g: (Super-)potential of (non-compact) source f.

•There's no need to worry about homogeneous solutions.

•Analytic closed form expressions of all the necessary super-potentials are available up to 2.5 PN order inclusively and 3.5 PN order.

• At 3 PN order, we could not find all. We instead find the potentials in the neighborhood of the body zone, which are what we need to evaluate surface integrals to derive EOM, or change the order of integrations: compute surface integral first and then compute remaining Poisson integral.

Equations of motion

1. Conservation law
2. Surface integral approach
3. Mass-Energy relation
4. Momentum-velocity relation
5. General form of equations of motion

Conservation law and surface integral approach

Separate Theta part and chi part

$$\begin{aligned}
P^{\mu}_{A}(\tau) &= \epsilon^{2}\int_{B_A} d^{3}\alpha_{A}\Lambda^{\tau\mu}_{N} \\
&= \epsilon^{2}\int_{B_A} d^{3}\alpha_{A}\Theta^{\tau\mu}_{N} \\
&+ \epsilon^{-4}\int_{\partial B_A} \frac{dS_k}{16\pi}(h^{\mu k}h^{\tau\alpha} - h^{\tau\tau}h^{k\alpha})_{,\alpha} \\
&\equiv P^{\mu}_{\Theta A}(\tau) + P^{\mu}_{\chi A}(\tau)
\end{aligned}$$

Conservation law:

$$\Lambda^{\mu\nu}{}_{,\nu} = 0, \Theta^{\mu\nu}{}_{,\nu} = 0, \chi^{\mu\nu\alpha\beta}{}_{,\alpha\beta\nu} = 0.$$

Surface integral form for evolution equation of 4-momentum as a result of energy-momentum conservation:

$$\frac{dP^{\mu}_{\Theta A}}{d\tau} = -\epsilon^{-4}\oint_{\partial B_A} dS_k \Theta^{k\mu}_{N} + \epsilon^{-4}v^{k}_{A}\oint_{\partial B_A} dS_k \Theta^{\tau\mu}_{N}.$$

Mass Energy relation

$$\frac{dP^{\mu}_{\Theta A}}{d\tau} = -\epsilon^{-4}\oint_{\partial B_A} dS_k \Theta^{k\mu}_N + \epsilon^{-4} v^k_A \oint_{\partial B_A} dS_k \Theta^{\tau\mu}_N .$$

$$h^{\tau\tau} = \sum_A \frac{4P^{\tau}_A}{r_A} + \cdots .$$

$$\begin{aligned} \frac{dP^{\tau}_{\Theta 1}}{d\tau} &= -\epsilon^{-4}\oint_{\partial B_1} dS_k \Theta^{k\tau}_N + \epsilon^{-4} v^k_1 \oint_{\partial B_1} dS_k \Theta^{\tau\tau}_N \\ &= -\epsilon^2 \frac{m_1 m_2}{r_{12}^2}\left[4(\vec{n}_{12}\cdot\vec{v}_1) - 3(\vec{n}_{12}\cdot\vec{v}_2)\right] + \cdots \end{aligned}$$

Integrate this equation functionally as

$$P^{\tau}_{\Theta A=1} = m_1\left[1 + \epsilon^2\left(\frac{1}{2}v_1^2 + \frac{3m_2}{r_{12}}\right)\right] + \cdots$$

Mass is defined as a integration constant, and independent of epsilon and time.

$$m_A \equiv \lim_{\epsilon\to 0} P^{\tau}_{\Theta A} \longleftarrow P^{\mu}_{\Theta A} = \int d^3\alpha_A \epsilon^2 \Theta^{\tau\mu} .$$

NB: 1) when epsilon is zero, there's no motion, no companion star. So This mass is defined on the rest frame of the star.

2) if body zone were extended to spatial infinity, this mass would become ADM mass of the star A (since epsilon → zero, there's no companion star).

3.5 PN evolution equation for energy.

$$\left(\frac{dP^\tau_{1\Theta}}{d\tau}\right) = \boxed{-\epsilon^2\frac{m_1m_2}{r_{12}^2}\left[4(\vec{n}_{12}\cdot\vec{v}_1)-3(\vec{n}_{12}\cdot\vec{v}_2)\right]}$$

1PN

$$+\epsilon^4\frac{m_1m_2}{r_{12}^2}\left[-\frac{9}{2}(\vec{n}_{12}\cdot\vec{v}_2)^3+\frac{1}{2}v_1^2(\vec{n}_{12}\cdot\vec{v}_2)+6(\vec{n}_{12}\cdot\vec{v}_1)(\vec{n}_{12}\cdot\vec{v}_2)^2\right.$$
$$-2v_1^2(\vec{n}_{12}\cdot\vec{v}_1)+4(\vec{v}_1\cdot\vec{v}_2)(\vec{n}_{12}\cdot\vec{V})+5v_2^2(\vec{n}_{12}\cdot\vec{v}_2)-4v_2^2(\vec{n}_{12}\cdot\vec{v}_1)$$
$$\left.+\frac{m_1}{r_{12}}(-4(\vec{n}_{12}\cdot\vec{v}_2)+6(\vec{n}_{12}\cdot\vec{v}_1))+\frac{m_2}{r_{12}}(-10(\vec{n}_{12}\cdot\vec{v}_1)+11(\vec{n}_{12}\cdot\vec{v}_2))\right]$$

2PN

$$+\epsilon^6\frac{m_1m_2}{r_{12}}\left[-\left(\frac{3}{2}v_1^4+2v_1^2v_2^2+4v_2^4\right)(\vec{n}_{12}\cdot\vec{v}_1)+\left(\frac{5}{8}v_1^4+\frac{3}{2}v_1^2v_2^2+7v_2^4\right)(\vec{n}_{12}\cdot\vec{v}_2)\right.$$
$$+\left(2v_1^2+4v_2^2\right)(\vec{n}_{12}\cdot\vec{v}_1)(\vec{v}_1\cdot\vec{v}_2)-\left(2v_1^2+8v_2^2\right)(\vec{n}_{12}\cdot\vec{v}_2)(\vec{v}_1\cdot\vec{v}_2)$$
$$+\left(3v_1^2+12v_2^2\right)(\vec{n}_{12}\cdot\vec{v}_1)(\vec{n}_{12}\cdot\vec{v}_2)^2-\left(\frac{3}{4}v_1^2+12v_2^2\right)(\vec{n}_{12}\cdot\vec{v}_2)^3$$
$$+2(\vec{n}_{12}\cdot\vec{v}_2)(\vec{v}_1\cdot\vec{v}_2)^2-6(\vec{n}_{12}\cdot\vec{v}_1)(\vec{n}_{12}\cdot\vec{v}_2)^2(\vec{v}_1\cdot\vec{v}_2)+6(\vec{n}_{12}\cdot\vec{v}_2)^3(\vec{v}_1\cdot\vec{v}_2)$$
$$-\frac{15}{2}(\vec{n}_{12}\cdot\vec{v}_1)(\vec{n}_{12}\cdot\vec{v}_2)^4-\frac{45}{8}(\vec{n}_{12}\cdot\vec{v}_2)^5$$
$$+\frac{m_1}{r_{12}}\left(\left(-42v_1^2-\frac{117}{4}v_2^2\right)(\vec{n}_{12}\cdot\vec{v}_1)+60(\vec{n}_{12}\cdot\vec{v}_1)^3\right.$$
$$+\left(\frac{137}{4}v_1^2+\frac{37}{2}v_2^2\right)(\vec{n}_{12}\cdot\vec{v}_2)$$
$$+\frac{297}{4}(\vec{n}_{12}\cdot\vec{v}_1)(\vec{v}.\vec{v}_2)-\frac{219}{4}(\vec{n}_{12}\cdot\vec{v}_2)(\vec{v}.\vec{v}_2)-151(\vec{n}_{12}\cdot\vec{v}_1)^2(\vec{n}_{12}\cdot\vec{v}_2)$$
$$\left.+109(\vec{n}_{12}\cdot\vec{v}_1)(\vec{n}_{12}\cdot\vec{v}_2)^2-23(\vec{n}_{12}\cdot\vec{v}_2)^3\right)$$
$$+\frac{m_2}{r_{12}}\left(-\left(13v_1^2+18v_2^2\right)(\vec{n}_{12}\cdot\vec{v}_1)+\left(17v_1^2+25v_2^2\right)(\vec{n}_{12}\cdot\vec{v}_2)\right.$$
$$+26(\vec{n}_{12}\cdot\vec{v}_1)(\vec{v}_1\cdot\vec{v}_2)-28(\vec{n}_{12}\cdot\vec{v}_2)(\vec{v}_1\cdot\vec{v}_2)+2(\vec{n}_{12}\cdot\vec{v}_1)^2(\vec{n}_{12}\cdot\vec{v}_2)$$
$$\left.+16(\vec{n}_{12}\cdot\vec{v}_1)(\vec{n}_{12}\cdot\vec{v}_2)^2-20(\vec{n}_{12}\cdot\vec{v}_2)^3\right)$$
$$+\frac{m_1^2}{r_{12}^2}\left(\frac{33}{4}(\vec{n}_{12}\cdot\vec{v}_1)-\frac{13}{2}(\vec{n}_{12}\cdot\vec{v}_2)\right)-\frac{m_1m_2}{r_{12}^2}\left(\frac{35}{4}(\vec{n}_{12}\cdot\vec{v}_1)+\frac{17}{4}(\vec{n}_{12}\cdot\vec{v}_2)\right)$$
$$\left.+\frac{m_2^2}{r_{12}^2}\left(-12(\vec{n}_{12}\cdot\vec{v}_1)+\frac{23}{2}(\vec{n}_{12}\cdot\vec{v}_2)\right)\right].$$

3PN

$$+\frac{8m_1{}^4m_2}{15r_{12}{}^5}+\frac{56m_1{}^3m_2{}^2}{15r_{12}{}^5}+\frac{16m_1{}^2m_2{}^3}{5r_{12}{}^5}-\frac{m_1^2m_2^2}{r_{12}^4}\left[\frac{148}{15}v_1{}^2+\frac{392}{15}(\vec{n}_{12}\cdot\vec{v}_1)^2+\frac{184}{15}(\vec{n}_{12}\cdot\vec{v}_2)^2\right.$$
$$\left.-\frac{52}{15}v_2{}^2-\frac{192}{5}(\vec{n}_{12}\cdot\vec{v}_1)(\vec{n}_{12}\cdot\vec{v}_2)+\frac{40}{3}(\vec{v}_1\cdot\vec{v}_2)\right]+\frac{m_1^3m_2}{r_{12}^4}\left[-\frac{16}{5}(\vec{n}_{12}\cdot\vec{v}_1)^2+\frac{8}{5}(\vec{n}_{12}\cdot\vec{v}_2)^2+\frac{4}{5}v_1{}^2\right.$$
$$\left.-\frac{4}{5}v_2{}^2+\frac{8}{5}(\vec{n}_{12}\cdot\vec{v}_1)(\vec{n}_{12}\cdot\vec{v}_2)\right]+\frac{m_1^2m_2}{r_{12}^3}\left[-\frac{8}{15}v_1{}^4+\frac{4}{15}v_2{}^4+\frac{8}{5}(\vec{n}_{12}\cdot\vec{v}_1)^2v_1{}^2-\frac{4}{5}(\vec{n}_{12}\cdot\vec{v}_2)^2v_1{}^2\right.$$
$$-\frac{4}{5}(\vec{n}_{12}\cdot\vec{v}_1)(\vec{n}_{12}\cdot\vec{v}_2)v_1{}^2-\frac{8}{15}(\vec{v}_1\cdot\vec{v}_2)^2+\frac{8}{5}(\vec{n}_{12}\cdot\vec{v}_1)^2v_2{}^2-\frac{4}{5}(\vec{n}_{12}\cdot\vec{v}_2)^2v_2{}^2-\frac{4}{15}v_1{}^2v_2{}^2$$
$$-\frac{4}{5}(\vec{n}_{12}\cdot\vec{v}_1)(\vec{n}_{12}\cdot\vec{v}_2)v_2{}^2-\frac{4}{15}(\vec{v}_1\cdot\vec{v}_2)v_2{}^2-\frac{16}{5}(\vec{n}_{12}\cdot\vec{v}_1)^2(\vec{v}_1\cdot\vec{v}_2)+\frac{8}{5}(\vec{n}_{12}\cdot\vec{v}_2)^2(\vec{v}_1\cdot\vec{v}_2)$$
$$\left.+\frac{4}{3}v_1{}^2(\vec{v}_1\cdot\vec{v}_2)+\frac{8}{5}(\vec{n}_{12}\cdot\vec{v}_1)(\vec{n}_{12}\cdot\vec{v}_2)(\vec{v}_1\cdot\vec{v}_2)\right].$$

3.5PN

We can integrate this!

$$P^\tau_{\Theta A}(\tau)=\epsilon^2\int_{B_A}d^3\alpha_A(-g)\left(T^{\tau\tau}_A+t^{\tau\tau}_{LL}\right)$$

Tensor density of weitht -2, not -1.

$$m_A\equiv\lim_{\epsilon\to 0}P^\tau_{\Theta A}$$

3.5 PN mass-energy relation.

$$P^{\tau}_{1\Theta} = m_1 \sum_{k=0}^{7} \epsilon^k \, {}_k\Gamma_1 + O(\epsilon^8).$$

$${}_2\Gamma_1 = \frac{1}{2}v_1^2 + \frac{3m_2}{r_{12}},$$

1PN

$${}_4\Gamma_1 = -\frac{3m_2}{2r_{12}}(\vec{n}_{12}\cdot\vec{v}_2)^2 + \frac{2m_2}{r_{12}}v_2^2 + \frac{7m_2}{2r_{12}}v_1^2 - \frac{4m_2}{r_{12}}(\vec{v}_1\cdot\vec{v}_2) + \frac{3}{8}v_1^4 + \frac{7m_2^2}{2r_{12}^2} - \frac{5m_1m_2}{2r_{12}^2}.$$

2PN

$$\begin{aligned}
{}_6\Gamma_1 = {} & \frac{m_1^2m_2}{2r_{12}^3} + \frac{21m_1m_2^2}{4r_{12}^3} + \frac{5m_2^3}{2r_{12}^3} + \frac{5}{16}v_1^6 \\
& + \frac{m_2^2}{r_{12}^2}\left(\frac{45}{4}v_1^2 + \frac{19}{2}v_2^2 + \frac{1}{2}(\vec{n}_{12}\cdot\vec{v}_1)^2 - 19(\vec{v}_1\cdot\vec{v}_2) - (\vec{n}_{12}\cdot\vec{v}_1)(\vec{n}_{12}\cdot\vec{v}_2) - 3(\vec{n}_{12}\cdot\vec{v}_2)^2\right) \\
& + \frac{m_1m_2}{r_{12}^2}\left(\frac{43}{8}v_1^2 + \frac{53}{8}v_2^2 - \frac{69}{8}(\vec{n}_{12}\cdot\vec{v}_1)^2 - \frac{53}{4}(\vec{v}_1\cdot\vec{v}_2) + \frac{85}{4}(\vec{n}_{12}\cdot\vec{v}_1)(\vec{n}_{12}\cdot\vec{v}_2)\right. \\
& \left. - \frac{69}{8}(\vec{n}_{12}\cdot\vec{v}_2)^2\right) + \frac{m_2}{r_{12}}\left(\frac{33}{8}v_1^4 + \frac{3}{2}v_1^2v_2^2 + v_2^4 - 6v_1^2(\vec{v}_1\cdot\vec{v}_2) - 4v_2^2(\vec{v}_1\cdot\vec{v}_2)\right. \\
& - \frac{7}{4}v_1^2(\vec{n}_{12}\cdot\vec{v}_2)^2 - \frac{5}{2}v_2^2(\vec{n}_{12}\cdot\vec{v}_2)^2 + 2(\vec{v}_1\cdot\vec{v}_2)^2
\end{aligned}$$

3PN

$$\begin{aligned}
{}_7\Gamma_1 = {} & -\frac{8{m_1}^2m_2(\vec{n}_{12}\cdot\vec{V})}{15{r_{12}}^3} + \frac{4m_1m_2(\vec{n}_{12}\cdot\vec{V})V^2}{15{r_{12}}^2} \\
& - \frac{16m_1{m_2}^2(\vec{n}_{12}\cdot\vec{V})}{5{r_{12}}^3},
\end{aligned}$$

3.5PN

$$P^{\tau}_{\Theta A} = m_A[\sqrt{-g}u^{\tau}_A]^{HPF}$$

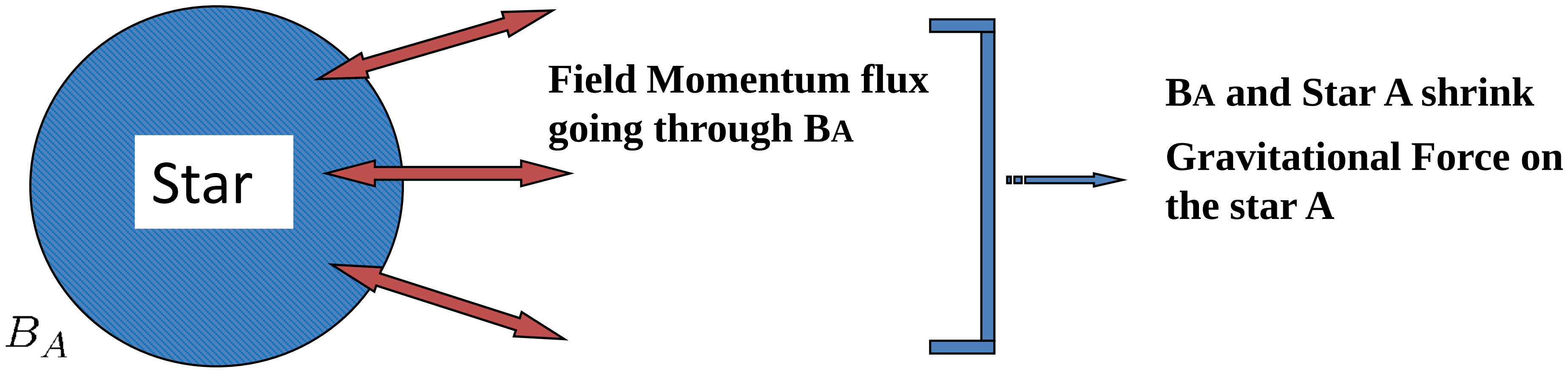
HPF: Hadamard Partie Finie.

- **We need 2.5 PN field to derive 3.5 PN mass-energy relation.**

General form of equations of motion

$$
\begin{aligned}
m_A \frac{dv_A^i}{d\tau} &= -\epsilon^{-4} \oint_{\partial B_A} dS_k \Theta_N^{ki} + \epsilon^{-4} v_A^k \oint_{\partial B_A} dS_k \Theta_N^{\tau i} \\
&\quad + \epsilon^{-4} v_A^i \left(\oint_{\partial B_A} dS_k \Theta_N^{k\tau} - v_A^k \oint_{\partial B_A} dS_k \Theta_N^{\tau\tau} \right) \\
&\quad - \frac{dQ_{\Theta A}^i}{d\tau} - \epsilon^2 \frac{d^2 D_{\Theta A}^i}{d\tau^2} + (m_A - P_{\Theta A}^\tau) \frac{dv_A^i}{d\tau}
\end{aligned}
$$

The general form of the equation of motion (Itoh, Futamase & Asada (2000))



3.5 PN EOM (monopole terms)

1PN

$$
\begin{aligned}
m_1\frac{dv_1^i}{d\tau} = & -\frac{m_1 m_2}{r_{12}^2}n_{12}^i \\
& +\epsilon^2\frac{m_1 m_2}{r_{12}^2}n_{12}^i\left[-v_1^2-2v_2^2+4(\vec{v}_1\cdot\vec{v}_2)+\frac{3}{2}(\vec{n}_{12}\cdot\vec{v}_2)^2+\frac{5m_1}{r_{12}}+\frac{4m_2}{r_{12}}\right] \\
& +\epsilon^2\frac{m_1 m_2}{r_{12}^2}V^i\left[4(\vec{n}_{12}\cdot\vec{v}_1)-3(\vec{n}_{12}\cdot\vec{v}_2)\right]
\end{aligned}
$$

2PN

$$
\begin{aligned}
& +\epsilon^4\frac{m_1 m_2}{r_{12}^2}n_{12}^i\Big[-2v_2^4+4v_2^2(\vec{v}_1\cdot\vec{v}_2)-2(\vec{v}_1\cdot\vec{v}_2)^2+\frac{3}{2}v_1^2(\vec{n}_{12}\cdot\vec{v}_2)^2+\frac{9}{2}v_2^2(\vec{n}_{12}\cdot\vec{v}_2)^2 \\
& \quad -6(\vec{v}_1\cdot\vec{v}_2)(\vec{n}_{12}\cdot\vec{v}_2)^2-\frac{15}{8}(\vec{n}_{12}\cdot\vec{v}_2)^4-\frac{57}{4}\frac{m_1^2}{r_{12}^2}-9\frac{m_2^2}{r_{12}^2}-\frac{69}{2}\frac{m_1 m_2}{r_{12}^2} \\
& \quad +\frac{m_1}{r_{12}}\left(-\frac{15}{4}v_1^2+\frac{5}{4}v_2^2-\frac{5}{2}(\vec{v}_1\cdot\vec{v}_2)+\frac{39}{2}(\vec{n}_{12}\cdot\vec{v}_1)^2\right. \\
& \qquad \left. - -39(\vec{n}_{12}\cdot\vec{v}_1)(\vec{n}_{12}\cdot\vec{v}_2)+\frac{17}{2}(\vec{n}_{12}\cdot\vec{v}_2)^2\right) \\
& \quad +\frac{m_2}{r_{12}}\left(4v_2^2-8(\vec{v}_1\cdot\vec{v}_2)+2(\vec{n}_{12}\cdot\vec{v}_1)^2-4(\vec{n}_{12}\cdot\vec{v}_1)(\vec{n}_{12}\cdot\vec{v}_2)-6(\vec{n}_{12}\cdot\vec{v}_2)^2\right)\Big] \\
& +\epsilon^4\frac{m_1 m_2}{r_{12}^2}V^i\Big[\frac{m_1}{r_{12}}\left(-\frac{63}{4}(\vec{n}_{12}\cdot\vec{v}_1)+\frac{55}{4}(\vec{n}_{12}\cdot\vec{v}_2)\right)+\frac{m_2}{r_{12}}\left(-2(\vec{n}_{12}\cdot\vec{v}_1)-2(\vec{n}_{12}\cdot\vec{v}_2)\right) \\
& \quad +v_1^2(\vec{n}_{12}\cdot\vec{v}_2)+4v_2^2(\vec{n}_{12}\cdot\vec{v}_1)-5v_2^2(\vec{n}_{12}\cdot\vec{v}_2)-4(\vec{v}_1\cdot\vec{v}_2)(\vec{n}_{12}\cdot\vec{V}) \\
& \quad -6(\vec{n}_{12}\cdot\vec{v}_1)(\vec{n}_{12}\cdot\vec{v}_2)^2+\frac{9}{2}(\vec{n}_{12}\cdot\vec{v}_2)^3\Big]
\end{aligned}
$$

2.5PN

$$
+\epsilon^5\frac{4m_1^2 m_2}{5r_{12}^3}\left[n_{12}^i(\vec{n}_{12}\cdot\vec{V})\left(-6\frac{m_1}{r_{12}}+\frac{52}{3}\frac{m_2}{r_{12}}+3V^2\right)+V^i\left(2\frac{m_1}{r_{12}}-8\frac{m_2}{r_{12}}-V^2\right)\right]
$$

3PN

$$
\begin{aligned}
& +\epsilon^6\frac{m_1 m_2}{r_{12}^2}n_{12}^i\Big[\frac{35}{16}(\vec{n}_{12}\cdot\vec{v}_2)^6-\frac{15}{8}(\vec{n}_{12}\cdot\vec{v}_2)^4v_1^2+\frac{15}{2}(\vec{n}_{12}\cdot\vec{v}_2)^4(\vec{v}_1\cdot\vec{v}_2) \\
& \quad +3(\vec{n}_{12}\cdot\vec{v}_2)^2(\vec{v}_1\cdot\vec{v}_2)^2-\frac{15}{2}(\vec{n}_{12}\cdot\vec{v}_2)^4v_2^2+\frac{3}{2}(\vec{n}_{12}\cdot\vec{v}_2)^2v_1^2v_2^2 \\
& \quad -12(\vec{n}_{12}\cdot\vec{v}_2)^2(\vec{v}_1\cdot\vec{v}_2)v_2^2-2(\vec{v}_1\cdot\vec{v}_2)^2v_2^2+\frac{15}{2}(\vec{n}_{12}\cdot\vec{v}_2)^2v_2^4 \\
& \quad +4(\vec{v}_1\cdot\vec{v}_2)v_2^4-2v_2^6 \\
& \quad +\frac{m_1}{r_{12}}\Big(-\frac{171}{8}(\vec{n}_{12}\cdot\vec{v}_1)^4+\frac{171}{2}(\vec{n}_{12}\cdot\vec{v}_1)^3(\vec{n}_{12}\cdot\vec{v}_2) \\
& \qquad -\frac{723}{4}(\vec{n}_{12}\cdot\vec{v}_1)^2(\vec{n}_{12}\cdot\vec{v}_2)^2+\frac{383}{2}(\vec{n}_{12}\cdot\vec{v}_1)(\vec{n}_{12}\cdot\vec{v}_2)^3 \\
& \qquad -\frac{455}{8}(\vec{n}_{12}\cdot\vec{v}_2)^4+\frac{229}{4}(\vec{n}_{12}\cdot\vec{v}_1)^2v_1^2-\frac{205}{2}(\vec{n}_{12}\cdot\vec{v}_1)(\vec{n}_{12}\cdot\vec{v}_2)v_1^2 \\
& \qquad +\frac{191}{4}(\vec{n}_{12}\cdot\vec{v}_2)^2v_1^2-\frac{91}{8}v_1^4-\frac{229}{2}(\vec{n}_{12}\cdot\vec{v}_1)^2(\vec{v}_1\cdot\vec{v}_2) \\
& \qquad +244(\vec{n}_{12}\cdot\vec{v}_1)(\vec{n}_{12}\cdot\vec{v}_2)(\vec{v}_1\cdot\vec{v}_2)-\frac{225}{2}(\vec{n}_{12}\cdot\vec{v}_2)^2(\vec{v}_1\cdot\vec{v}_2) \\
& \qquad +\frac{91}{2}v_1^2(\vec{v}_1\cdot\vec{v}_2)-\frac{177}{4}(\vec{v}_1\cdot\vec{v}_2)^2+\frac{229}{4}(\vec{n}_{12}\cdot\vec{v}_1)^2v_2^2 \\
& \qquad -\frac{283}{2}(\vec{n}_{12}\cdot\vec{v}_1)(\vec{n}_{12}\cdot\vec{v}_2)v_2^2+\frac{259}{4}(\vec{n}_{12}\cdot\vec{v}_2)^2v_2^2-\frac{91}{4}v_1^2v_2^2 \\
& \qquad +43(\vec{v}_1\cdot\vec{v}_2)v_2^2-\frac{81}{8}v_2^4\Big) \\
& \quad +\frac{m_2}{r_{12}}\Big(-6(\vec{n}_{12}\cdot\vec{v}_1)^2(\vec{n}_{12}\cdot\vec{v}_2)^2+12(\vec{n}_{12}\cdot\vec{v}_1)(\vec{n}_{12}\cdot\vec{v}_2)^3 \\
& \qquad +6(\vec{n}_{12}\cdot\vec{v}_2)^4+4(\vec{n}_{12}\cdot\vec{v}_1)(\vec{n}_{12}\cdot\vec{v}_2)(\vec{v}_1\cdot\vec{v}_2) \\
& \qquad +12(\vec{n}_{12}\cdot\vec{v}_2)^2(\vec{v}_1\cdot\vec{v}_2)+4(\vec{v}_1\cdot\vec{v}_2)^2 \\
& \qquad -4(\vec{n}_{12}\cdot\vec{v}_1)(\vec{n}_{12}\cdot\vec{v}_2)v_2^2-12(\vec{n}_{12}\cdot\vec{v}_2)^2v_2^2-8(\vec{v}_1\cdot\vec{v}_2)v_2^2+4v_2^4\Big) \\
& \quad +\frac{m_2^2}{r_{12}^2}\Big(-(\vec{n}_{12}\cdot\vec{v}_1)^2+2(\vec{n}_{12}\cdot\vec{v}_1)(\vec{n}_{12}\cdot\vec{v}_2)+\frac{43}{2}(\vec{n}_{12}\cdot\vec{v}_2)^2 \\
& \qquad +18(\vec{v}_1\cdot\vec{v}_2)-9v_2^2\Big) \\
& \quad +\frac{m_1 m_2}{r_{12}^2}\Big(\frac{415}{8}(\vec{n}_{12}\cdot\vec{v}_1)^2-\frac{375}{4}(\vec{n}_{12}\cdot\vec{v}_1)(\vec{n}_{12}\cdot\vec{v}_2)+\frac{1113}{8}(\vec{n}_{12}\cdot\vec{v}_2)^2 \\
& \qquad -\frac{615\pi^2}{64}(\vec{n}_{12}\cdot\vec{V})^2+18v_1^2+\frac{123\pi^2}{64}V^2+33(\vec{v}_1\cdot\vec{v}_2)-\frac{33}{2}v_2^2\Big) \\
& \quad +\frac{m_1^2}{r_{12}^2}\Big(-\frac{2069}{8}(\vec{n}_{12}\cdot\vec{v}_1)^2+543(\vec{n}_{12}\cdot\vec{v}_1)(\vec{n}_{12}\cdot\vec{v}_2)-\frac{939}{4}(\vec{n}_{12}\cdot\vec{v}_2)^2 \\
& \qquad +\frac{471}{8}v_1^2-\frac{357}{4}(\vec{v}_1\cdot\vec{v}_2)+\frac{357}{8}v_2^2\Big) \\
& \quad +\frac{16m_2^3}{r_{12}^3}+\frac{m_1^2 m_2}{r_{12}^3}\left(\frac{547}{3}-\frac{41\pi^2}{16}\right)-\frac{13m_1^3}{12r_{12}^3}+\frac{m_1 m_2^2}{r_{12}^3}\left(\frac{545}{3}-\frac{41\pi^2}{16}\right)\Big] \\
& +\epsilon^6\frac{m_1 m_2}{r_{12}^2}V^i\Big[\frac{15}{2}(\vec{n}_{12}\cdot\vec{v}_1)(\vec{n}_{12}\cdot\vec{v}_2)^4-\frac{45}{8}(\vec{n}_{12}\cdot\vec{v}_2)^5-\frac{3}{2}(\vec{n}_{12}\cdot\vec{v}_2)^3v_1^2 \\
& \quad +6(\vec{n}_{12}\cdot\vec{v}_1)(\vec{n}_{12}\cdot\vec{v}_2)^2(\vec{v}_1\cdot\vec{v}_2)-6(\vec{n}_{12}\cdot\vec{v}_2)^3(\vec{v}_1\cdot\vec{v}_2) \\
& \quad -2(\vec{n}_{12}\cdot\vec{v}_2)(\vec{v}_1\cdot\vec{v}_2)^2-12(\vec{n}_{12}\cdot\vec{v}_1)(\vec{n}_{12}\cdot\vec{v}_2)^2v_2^2+12(\vec{n}_{12}\cdot\vec{v}_2)^3v_2^2 \\
& \quad +(\vec{n}_{12}\cdot\vec{v}_2)v_1^2v_2^2-4(\vec{n}_{12}\cdot\vec{v}_1)(\vec{v}_1\cdot\vec{v}_2)v_2^2+8(\vec{n}_{12}\cdot\vec{v}_2)(\vec{v}_1\cdot\vec{v}_2)v_2^2 \\
& \quad +4(\vec{n}_{12}\cdot\vec{v}_1)v_2^4-7(\vec{n}_{12}\cdot\vec{v}_2)v_2^4 \\
& \quad +\frac{m_2}{r_{12}}\Big(-2(\vec{n}_{12}\cdot\vec{v}_1)^2(\vec{n}_{12}\cdot\vec{v}_2)+8(\vec{n}_{12}\cdot\vec{v}_1)(\vec{n}_{12}\cdot\vec{v}_2)^2+2(\vec{n}_{12}\cdot\vec{v}_2)^3 \\
& \qquad +2(\vec{n}_{12}\cdot\vec{v}_1)(\vec{v}_1\cdot\vec{v}_2)+4(\vec{n}_{12}\cdot\vec{v}_2)(\vec{v}_1\cdot\vec{v}_2) \\
& \qquad -2(\vec{n}_{12}\cdot\vec{v}_1)v_2^2-4(\vec{n}_{12}\cdot\vec{v}_2)v_2^2\Big) \\
& \quad +\frac{m_1}{r_{12}}\Big(-\frac{243}{4}(\vec{n}_{12}\cdot\vec{v}_1)^3+\frac{565}{4}(\vec{n}_{12}\cdot\vec{v}_1)^2(\vec{n}_{12}\cdot\vec{v}_2) \\
& \qquad -\frac{269}{4}(\vec{n}_{12}\cdot\vec{v}_1)(\vec{n}_{12}\cdot\vec{v}_2)^2-\frac{95}{12}(\vec{n}_{12}\cdot\vec{v}_2)^3+\frac{207}{8}(\vec{n}_{12}\cdot\vec{v}_1)v_1^2 \\
& \qquad -\frac{137}{8}(\vec{n}_{12}\cdot\vec{v}_2)v_1^2-36(\vec{n}_{12}\cdot\vec{v}_1)(\vec{v}_1\cdot\vec{v}_2)+\frac{27}{4}(\vec{n}_{12}\cdot\vec{v}_2)(\vec{v}_1\cdot\vec{v}_2) \\
& \qquad +\frac{81}{8}(\vec{n}_{12}\cdot\vec{v}_1)v_2^2+\frac{83}{8}(\vec{n}_{12}\cdot\vec{v}_2)v_2^2\Big) \\
& \quad +\frac{m_2^2}{r_{12}^2}\left(4(\vec{n}_{12}\cdot\vec{v}_1)+5(\vec{n}_{12}\cdot\vec{v}_2)\right) \\
& \quad +\frac{m_1 m_2}{r_{12}^2}\left(-\frac{307}{8}(\vec{n}_{12}\cdot\vec{v}_1)+\frac{479}{8}(\vec{n}_{12}\cdot\vec{v}_2)+\frac{123\pi^2}{32}(\vec{n}_{12}\cdot\vec{V})\right) \\
& \quad +\frac{m_1^2}{r_{12}^2}\left(\frac{311}{4}(\vec{n}_{12}\cdot\vec{v}_1)-\frac{357}{4}(\vec{n}_{12}\cdot\vec{v}_2)\right)\Big]
\end{aligned}
$$

3.5PN

$$
\begin{aligned}
& +\frac{m_1^4 m_2}{r_{12}^5}\left[n_{12}^i\left\{\frac{3992}{105}(\vec{n}_{12}\cdot\vec{v}_1)-\frac{4328}{105}(\vec{n}_{12}\cdot\vec{v}_2)\right\}-\frac{184}{21}V^i\right]+\frac{m_1^3 m_2^2}{r_{12}^5}\Big[\frac{6224}{105}V^i \\
& +n_{12}^i\left\{\frac{2872}{21}(\vec{n}_{12}\cdot\vec{v}_2)-\frac{13576}{105}(\vec{n}_{12}\cdot\vec{v}_1)\right\}\Big]+\frac{m_1^3 m_2}{r_{12}^4}V^i\Big[-\frac{132}{35}v_1{}^2-\frac{48}{35}v_2{}^2+\frac{52}{15}(\vec{n}_{12}\cdot\vec{v}_1)^2 \\
& +\frac{152}{35}(\vec{v}_1\cdot\vec{v}_2)-\frac{56}{15}(\vec{n}_{12}\cdot\vec{v}_1)(\vec{n}_{12}\cdot\vec{v}_2)-\frac{44}{15}(\vec{n}_{12}\cdot\vec{v}_2)^2\Big]+\frac{m_1^3 m_2}{r_{12}^4}n_{12}^i\Big[-\frac{4888}{105}(\vec{n}_{12}\cdot\vec{v}_1)v_1{}^2 \\
& +\frac{5056}{105}(\vec{n}_{12}\cdot\vec{v}_2)v_1{}^2-\frac{1028}{21}v_2{}^2(\vec{n}_{12}\cdot\vec{v}_1)+48(\vec{n}_{12}\cdot\vec{v}_1)^3+\frac{5812}{105}v_2{}^2(\vec{n}_{12}\cdot\vec{v}_2)+\frac{2056}{21}(\vec{n}_{12}\cdot\vec{v}_1)(\vec{v}_1\cdot\vec{v}_2) \\
& -\frac{2224}{21}(\vec{n}_{12}\cdot\vec{v}_2)(\vec{v}_1\cdot\vec{v}_2)-\frac{696}{5}(\vec{n}_{12}\cdot\vec{v}_1)^2(\vec{n}_{12}\cdot\vec{v}_2)+\frac{744}{5}(\vec{n}_{12}\cdot\vec{v}_1)(\vec{n}_{12}\cdot\vec{v}_2)^2-\frac{288}{5}(\vec{n}_{12}\cdot\vec{v}_2)^3\Big] \\
& +\frac{m_1^2 m_2^3}{r_{12}^5}\left[\frac{6388}{105}V^i-\frac{3172}{21}(\vec{n}_{12}\cdot\vec{V})n_{12}^i\right]+\frac{m_1^2 m_2}{r_{12}^3}V^i\Big[\frac{334}{35}v_1{}^4+\frac{654}{35}v_2{}^2v_1{}^2-\frac{1336}{35}(\vec{v}_1\cdot\vec{v}_2)v_1{}^2+\frac{292}{35}v_2{}^4 \\
& -\frac{348}{5}(\vec{n}_{12}\cdot\vec{v}_1)^2V^2+60(\vec{n}_{12}\cdot\vec{V})^4-\frac{1252}{35}v_2{}^2(\vec{v}_1\cdot\vec{v}_2)+\frac{684}{5}(\vec{n}_{12}\cdot\vec{v}_1)(\vec{n}_{12}\cdot\vec{v}_2)V^2+\frac{1308}{35}(\vec{v}_1\cdot\vec{v}_2)^2 \\
& -66(\vec{n}_{12}\cdot\vec{v}_2)^2V^2\Big]+\frac{m_1^2 m_2}{r_{12}^3}n_{12}^i\Big[-\frac{246}{35}(\vec{n}_{12}\cdot\vec{V})v_1{}^4-\frac{534}{35}v_2{}^2(\vec{n}_{12}\cdot\vec{v}_1)v_1{}^2+\frac{90}{7}v_2{}^2(\vec{n}_{12}\cdot\vec{v}_2)v_1{}^2 \\
& +\frac{1068}{35}(\vec{n}_{12}\cdot\vec{v}_1)(\vec{v}_1\cdot\vec{v}_2)v_1{}^2-\frac{984}{35}(\vec{n}_{12}\cdot\vec{v}_2)(\vec{v}_1\cdot\vec{v}_2)v_1{}^2-\frac{204}{35}v_2{}^4(\vec{n}_{12}\cdot\vec{v}_1)+60(\vec{n}_{12}\cdot\vec{v}_1)^3V^2 \\
& -56(\vec{n}_{12}\cdot\vec{V})^5+\frac{24}{7}v_2{}^4(\vec{n}_{12}\cdot\vec{v}_2)+\frac{984}{35}v_2{}^2(\vec{n}_{12}\cdot\vec{v}_1)(\vec{v}_1\cdot\vec{v}_2)-\frac{732}{35}v_2{}^2(\vec{n}_{12}\cdot\vec{v}_2)(\vec{v}_1\cdot\vec{v}_2) \\
& -180(\vec{n}_{12}\cdot\vec{v}_1)^2(\vec{n}_{12}\cdot\vec{v}_2)V^2-\frac{1068}{35}(\vec{n}_{12}\cdot\vec{v}_1)(\vec{v}_1\cdot\vec{v}_2)^2+174(\vec{n}_{12}\cdot\vec{v}_1)(\vec{n}_{12}\cdot\vec{v}_2)^2V^2 \\
& +\frac{180}{7}(\vec{n}_{12}\cdot\vec{v}_2)(\vec{v}_1\cdot\vec{v}_2)^2-54V^2(\vec{n}_{12}\cdot\vec{v}_2)^3\Big]+\frac{m_1^2 m_2{}^2}{r_{12}^4}V^i\Big[-\frac{152}{21}v_1{}^2-\frac{1768}{105}v_2{}^2+\frac{454}{15}(\vec{n}_{12}\cdot\vec{v}_1)^2 \\
& +\frac{2864}{105}(\vec{v}_1\cdot\vec{v}_2)-\frac{372}{5}(\vec{n}_{12}\cdot\vec{v}_1)(\vec{n}_{12}\cdot\vec{v}_2)+\frac{854}{15}(\vec{n}_{12}\cdot\vec{v}_2)^2\Big]+\frac{m_1^2 m_2{}^2}{r_{12}^4}n_{12}^i\Big[\frac{1432}{35}(\vec{n}_{12}\cdot\vec{v}_1)v_2{}^2 \\
& -\frac{5752}{105}(\vec{n}_{12}\cdot\vec{v}_2)v_2{}^2-\frac{582}{5}(\vec{n}_{12}\cdot\vec{v}_1)^3+\frac{3568}{105}(\vec{n}_{12}\cdot\vec{V})v_1^2-\frac{2864}{35}(\vec{n}_{12}\cdot\vec{v}_1)(\vec{v}_1\cdot\vec{v}_2) \\
& +\frac{10048}{105}(\vec{n}_{12}\cdot\vec{v}_2)(\vec{v}_1\cdot\vec{v}_2)+\frac{1746}{5}(\vec{n}_{12}\cdot\vec{v}_1)^2(\vec{n}_{12}\cdot\vec{v}_2)-\frac{1954}{5}(\vec{n}_{12}\cdot\vec{v}_1)(\vec{n}_{12}\cdot\vec{v}_2)^2+158(\vec{n}_{12}\cdot\vec{v}_2)
\end{aligned}
$$

This EOM is Lorentz-invariant (perturbation sense), admits conserved energy (when excluding rad. reac.), and has no undetermined coeff. We also checked 3.5 PN harmonic condition.

Leading order SO, SS, QO coupling forces and spin precessions.

$$F^i_{1\text{SO}} = \epsilon^4 \frac{m_1}{r_{12}^3}\left[6(\vec{s}_2 \times \vec{n}_{12})\cdot \vec{V} n^i_{12} + 4\vec{s}_2 \times \vec{V} - 6\vec{s}_2 \times \vec{n}_{12}(\vec{n}_{12}\cdot\vec{V})\right]$$
$$+\epsilon^4 \frac{m_2}{r_{12}^3}\left[6(\vec{s}_1 \times \vec{n}_{12})\cdot \vec{V} n^i_{12} + 3\vec{s}_1 \times \vec{V} - 3\vec{s}_1 \times \vec{n}_{12}(\vec{n}_{12}\cdot\vec{V})\right],$$

$$F^i_{1\text{SS}} = \epsilon^6\left[-\frac{15M_1^{jk}M_2^{jl}r^k_{12}r^l_{12}r^i_{12}}{r^7_{12}} + \frac{3M_1^{jk}M_2^{jk}r^i_{12}}{r^5_{12}} - \frac{3M_1^{ij}M_2^{jk}r^k_{12}}{r^5_{12}} - \frac{3M_1^{jk}M_2^{ki}r^j_{12}}{r^5_{12}}\right]$$
$$= \epsilon^6\frac{1}{r^4_{12}}\left[15(\vec{n}_{12}\cdot\vec{s}_1)(\vec{n}_{12}\cdot\vec{s}_2)n^i_{12} - 3s^i_1(\vec{n}_{12}\cdot\vec{s}_2) - 3s^i_2(\vec{n}_{12}\cdot\vec{s}_1) - 3n^i_{12}(\vec{s}_1\cdot\vec{s}_2)\right],$$

$$\frac{dM^{ij}_A}{d\tau} = -2\epsilon^{-2}v^{[i}_A P^{j]}_A - 2\epsilon^{-2}R^{[ij]}_A. \Longleftrightarrow \quad \frac{d\vec{\mathcal{S}}_1}{d\tau} = \epsilon^2\frac{m_2}{r^2_{12}}\left[\left(2\vec{v}_2 - \frac{3}{2}\vec{v}_1\right)\times\vec{n}_{12}\right]\times\vec{\mathcal{S}}_1 + \mathcal{O}(\epsilon^3),$$

$$F^i_{1\text{QO}} = \epsilon^4\frac{3}{2r^4_{12}}\left(m_1 I_2^{\langle kl\rangle} + m_2 I_1^{\langle kl\rangle}\right)\left(2\delta^{il}n^k_{12} - 5n^i_{12}n^k_{12}n^l_{12}\right)$$

See Tagoshi, Ohashi & Owen (2001) for 1PN SO force.

3.5 PN monopole EOM in a quasi-circular orbit in the Center of Mass Frame.

$$\frac{dV^i}{d\tau} = -\Omega^2 r^i_{12} (+A^i_{RR})$$

$$m^2\Omega^2 = \gamma^3 \Bigg[1 + \gamma(-3+\nu) + \gamma^2\left(6 + \frac{41}{4}\nu + \nu^2\right) + \gamma^6\left(-10 + \left(\frac{-2375}{24} + \frac{41\pi^2}{64}\right)\nu + \frac{19}{2}\nu^2 + \nu^3\right)\Bigg]$$

where V^i is a relative velocity,

A_{RR} is the 2.5 PN + 3.5 PN radiation reaction acceleration.

$m = m_1 + m_2, \nu = m_1 m_2/m^2, \gamma = m/r_{12}$

3 PN monopole Conserved Energy in a quasi-circular orbit in CMF

$$
\begin{aligned}
E(x) \;=\; & -\frac{m\nu x}{2}\left[1+\left(-\frac{3}{4}-\frac{1}{12}\nu\right)x+\left(-\frac{27}{8}+\frac{19}{8}\nu-\frac{1}{24}\nu^2\right)x^2\right. \\
& \left.+\left(-\frac{675}{64}+\left\{\frac{34445}{576}-\frac{205\pi^2}{96}\right\}\nu-\frac{155}{96}\nu^2-\frac{35}{5184}\nu^3\right)x^3\right].
\end{aligned}
$$

$x=(m\Omega)^{3/2}$, Ω is the orbital angular frequency.

c.f. Blanchet and Faye (2000)

$$
\begin{aligned}
E_{BF}(x) \;=\; & -\frac{m\nu x}{2}\left[1+\left(-\frac{3}{4}-\frac{1}{12}\nu\right)x+\left(-\frac{27}{8}+\frac{19}{8}\nu-\frac{1}{24}\nu^2\right)x^2\right. \\
& \left.+\;\left(-\frac{675}{64}+\left\{\frac{209323}{4032}-\frac{205\pi^2}{96}-\underline{\frac{110}{9}\lambda}\right\}\nu-\frac{155}{96}\nu^2-\frac{35}{5184}\nu^3\right)x^3\right].
\end{aligned}
$$

Obtained physically the same results

Having physically the same results between these two means that a comparable mass binary follows a geodesic of "space-time" described by (dimensionally) regularized metric.

(2) Regularized geodesics or, regularized action **(Blanchet & Faye,….)**

$$[u^{\nu} u^{\mu}{}_{;\nu}]^{reg} = 0$$

(3)Surface Integral Approach: **(Einstein, Infeld & Hoffmann, YI, Futamase & Asada)**

$$F_1^i = -\oint_{\partial B_1} dS_j t^{ij},$$

$$t^{ij} \equiv \frac{1}{4\pi}\left(\frac{\partial\phi}{\partial x^i}\frac{\partial\phi}{\partial x^j} - \frac{\delta^{ij}}{2}\frac{\partial\phi}{\partial x^k}\frac{\partial\phi}{\partial x^k}\right).$$

self-force approach implication

To the linear order in the mass ratio, the metric perturbation about the background metric g obeys, in the harmonic condition $\nabla_a \bar{h}^{ab} = 0$,

$$\nabla^2 \bar{h}_{ab} + 2R_a{}^c{}_b{}^d \bar{h}_{cd} = -16\pi T_{ab}$$

Denote the inhomogeneous solution by $\bar{h}^{\mathrm{S}}_{ab}$, the homogeneous solution $\bar{h}^{\mathrm{R}}_{ab} \equiv \bar{h}^{\mathrm{ret}}_{ab} - \bar{h}^{\mathrm{S}}_{ab}$ actually gives the self-force. In other words, the lighter particle follows a geodesic of “space-time” augmented by the metric $g_{ab} + h^{\mathrm{R}}_{ab}$.

Detweiler & Whiting 2002

WAVEFORM

References

- Direct-Integration of Relaxed Einstein equations:
  - "Gravity: Newtonian, post-Newtonian, Relativistic": a text book by C. M. Will and E. Poisson
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  - L. Blanchet's article in Living reviews in relativity
- Introductory textbook
  - "Gravitational Waves I: Theory and experiments": a text book by M. Maggiore
- Review articles including Effective one body approach
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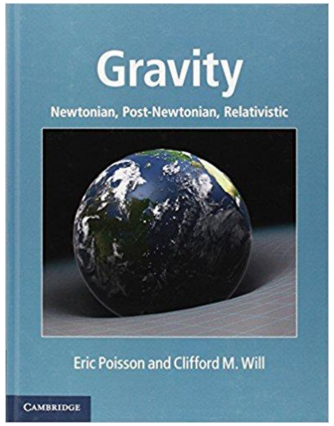
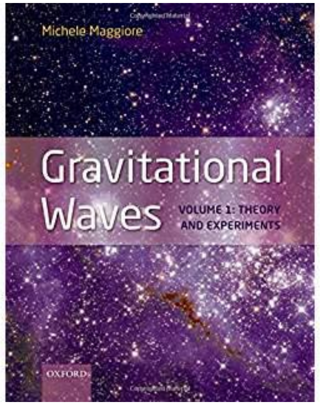
  - T. Damour, arxiv:1312.3505





Post-Newtonian approximation break-down

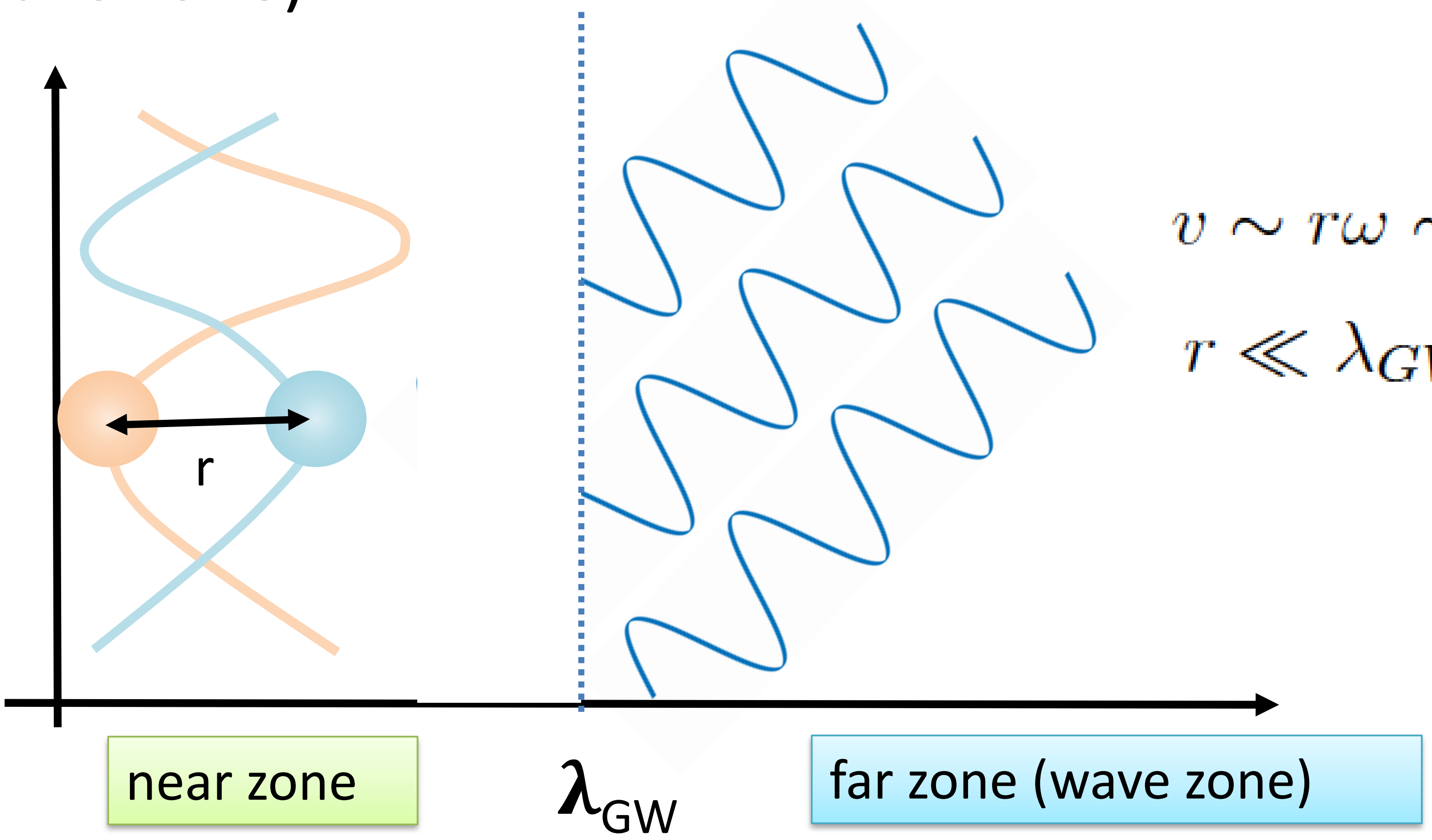
- The integrands of the formal solution of the relaxed Einstein equations are non-compact support.
- By slow motion expansion,

$$h(\tau, x^i) \sim \int d^3y \, \frac{f(\tau - \epsilon|\vec{x} - \vec{y}|, \vec{y})}{|\vec{x} - \vec{y}|} \sim \sum_n^{\infty} \frac{(-1)^n}{n!} \epsilon^n \int d^3y \, |\vec{x} - \vec{y}|^{n-1} \frac{d^n}{dt^n} f(t, \vec{y}).$$

- Hence whatever small ε is, the integrals diverge. This is an indication that PN expansion is an asymptotic expansion.

How to deal with PNA break-down

- Two methods have been proposed: DIRE & MPM
- Split the integral region into two zones: near zone and far zone (wave zone)



$$v \sim r\omega \sim r/\lambda_{GW} = \epsilon \ll 1$$

$$r \ll \lambda_{GW}$$

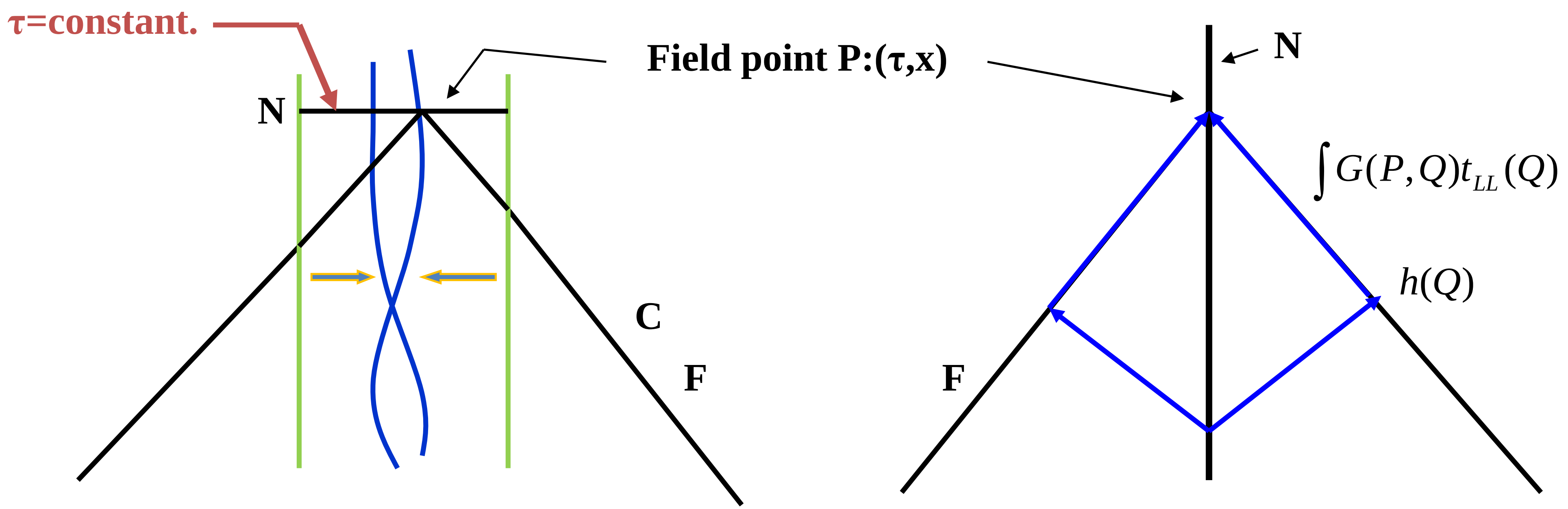
Direct Integration of Relaxed Einstein equations (DIRE) approach (Will-Wiseman-Pati)

- Split the integral region into two: near zone and far zone.

$$h^{\mu\nu}_{F(C)} = h^{\mu\nu}_{F(N)} + h^{\mu\nu}_{F(F)} + h^{\mu\nu}_{H},$$

$$h^{\mu\nu}_{F(N)}(\tau, \vec{x}) = 4\int_{N=\{y:|y|<\mathcal{R}/\epsilon\}} d^3y \frac{\Lambda^{\mu\nu}(\tau - \epsilon|\vec{x}-\vec{y}|, y^k; \epsilon)}{|\vec{x}-\vec{y}|}$$

$$h^{\mu\nu}_{F(C)}(t, \vec{x}) = 4\int_{F=\{y:|y|>\epsilon\mathcal{R}\}} d^3y \frac{\Lambda^{\mu\nu}(t - |\vec{x}-\vec{y}|, y^k; \epsilon)}{|\vec{x}-\vec{y}|}$$



DIRE approach

- Introduce the retarded time : $u = t - r$
- The near zone contribution to the far zone field is evaluated using the source multipole moments. These source multipole moments are functionals of mass, velocity, spins ….

$$
\begin{aligned}
h^{\mu\nu}_{F(N)} &= 4 \int_N \frac{d^3 y}{|\vec{x} - \vec{y}|} \Lambda^{\mu\nu}(t - |\vec{x} - \vec{y}|, \vec{y}) \\
&= 4 \sum_{l=0} \frac{(-1)^l}{l!} \partial_{K_l} \left(\frac{1}{r} M^{K_l \mu\nu}(u) \right)
\end{aligned}
$$

$$
M^{K_l \mu\nu}(u) \equiv \int_N d^3 y \, \Lambda^{\mu\nu}(u, \vec{y}) \, y^{K_l} .
$$

DIRE approach

- Far zone contribution is evaluated by directly computing the integrals.

$$h^{\mu\nu}_{F(F)}(t,\vec{x}) = 4\int_F d^3x' \, \frac{\Lambda^{\mu\nu}(t-|\vec{x}-\vec{x}'|,\vec{x}')}{|\vec{x}-\vec{x}'|}$$
$$= 4\int_{-\infty}^{u} du' \oint_F \frac{\Lambda^{\mu\nu}(u'+r',\vec{x}')}{t-u'-\vec{n}'\cdot\vec{x}} [r'(u',\Omega')]^2 \, d\Omega,$$

$$r'(u',\Omega') = \frac{(t-u')^2 - r^2}{2(t-u'-\vec{n}'\cdot\vec{x})}.$$

- Decompose the integrand into symmetric-trace-free tensor:

$$\Lambda^{\mu\nu} \sim f_{B,L} r^{-B} n^{\langle L\rangle}$$

DIRE approach

- The Integrands is a sums of terms that consists of the source multipole moments times some function independent of the system physical quantities both of which depend on u. Then integrate by parts, increasing the u-derivative of the source multipole moments up to the necessary PN order.

$$h^{\mu\nu}_{F(F)}(u,x^i) = \sum_{\substack{B\neq 2\\ l}} \left(\frac{2}{r}\right)^{B-2} n^{\langle L\rangle} \sum_{q=0} \mathcal{D}^q_{B,L}(z)\, r^q\, \frac{d^q f_{B,L}(u)}{du^q}$$
$$+\frac{n^{\langle L\rangle}}{r}\int_0^\infty f_{2,L}(u-s) Q_L\left(1+\frac{s}{r}\right) + n^{\langle L\rangle}\sum_{q=0} \mathcal{D}^q_{2,L}(z)\, r^q\, \frac{d^q f_{2,L}(u)}{du^q}$$

with $z = \mathcal{R}/r$. The coefficients $D^q_{B,L}(z)$ are evaluated using the Legendre Polynomials.

Multipolar Post-Minkowskian approach (MPM)

- There is no need to use the same coordinates for the far zone and the near zone.
- In the far zone, use the tortoise coordinates ("radiative coordinates" deviated little from the original harmonic ones) with the mass M_{ADM} of the system. This way, the Coulomb logarithmic phase shift can naturally be incorporated.
- Solve the Einstein equations iteratively.

$$\Box h^{\alpha\beta} = \Lambda^{\alpha\beta} = N^{\alpha\beta}(h,h) + M^{\alpha\beta}(h,h,h) + \mathcal{O}(h^4)$$

MPM solution in the wave zone (*T*,*R*)

STF moments

At the leading order: $\Box h^{\alpha\beta}_{(W,1)} = 0$

The general solutions to these equations: $u^{\alpha\beta}_{(W,1)} = \frac{1}{R}\sum_{\ell=0} \partial_{L_\ell}[K^{\alpha\beta}_{L_\ell}(T - R/c)]$

Adding functions (again solutions of homogenous wave equations) to satisfy the gauge condition.

$$h^{\alpha\beta}_{(W,1)} = u^{\alpha\beta}_{(W,1)} + v^{\alpha\beta}_{(W,1)}$$

$$v^{\alpha\beta}_{(W,1)} = \partial^\alpha \xi^\beta_{(W,1)} + \partial^\beta \xi^\alpha_{(W,1)} - \eta^{\alpha\beta}\partial_\rho \xi^\rho_{(W,1)}$$

In general, the solution depends on 6 SFT multipole moments $\{I_L, J_L, W_L, X_L, Y_L, Z_L\}$ which can be combined into two gauge independent moments $\{M_L, S_L\}$

MPM solution in the wave zone

At the next-to-leading order: $\Box h^{\alpha\beta}_{(W,2)} = R^B N^{\alpha\beta}(h_{(W,1)}, h_{(W,1)})$

Because $h_{(w,1)}$ is divergent at the origin, one multiplies regularization factor R^B (B: complex number and its real part is positive), solve the wave equations and then find its analytic continuation to B = 0.

Likewise, at a general order n: $\Box h^{\alpha\beta}_{(W,n)} = -16\pi R^B \Lambda^{\alpha\beta}_{(W,n)}$

Inhomogeneous solutions: $u^{\alpha\beta}_{(W,n)} = -16\pi \mathrm{FP}_{B=0}[\Box^{-1}_{ret} R^B \Lambda^{\alpha\beta}_{(W,n)}]$

Again we have gauge functions.

$$h^{\alpha\beta}_{(W,n)} = u^{\alpha\beta}_{(W,n)} + v^{\alpha\beta}_{(W,n)}$$

MPM solution in the near zone

In the near zone: $u^{\alpha\beta}_{(N,n)} = -16\pi \mathrm{FP}_{B=0}[\Delta^{-1}_{ret}(r^B(\Lambda^{\alpha\beta}_{(N,n)} - \partial^2_t h_{(N,n-2)}/16\pi)]$

Gauge functions: $v^{\alpha\beta}_{(N,n)} = \frac{1}{r}\sum_{\ell=0} \partial_{L_\ell}[F^{\alpha\beta}_{L_\ell}(t-r/c) - F^{\alpha\beta}_{L_\ell}(t+r/c)]$

Then at the overlapping zone, one matches the two solutions and finds the relationship between the source multipole moments defined in the near zone and the multipole moments $\{M_L , S_L\}$, and obtains the field at an observer.

$$h^{TT}_{ij} = \frac{4G}{c^2 R}\sum_{k,q}\mathcal{P}_{ijkq}(\vec{N})\sum_{\ell=2}^{\infty}\frac{1}{c^\ell \ell!}\left[N_{L_{\ell-2}}U_{kqL_{\ell-2}} - \frac{2\ell N_{mL_{\ell-2}}\epsilon_{mn(k}V_{q)nL_{\ell-2}}}{c(\ell+1)}\right]$$

The $\{U_L,V_L\}$ are functionals of the source canonical multipole moments $\{M_L,S_L\}$

$$U_L = \frac{d^\ell M_L}{dT^\ell} + \frac{2GM}{c^3}\int_0^\infty d\tau\, M_L^{(\ell+2)}(T_R-\tau)\left[\log\left(\frac{c\tau}{2r_0}\right)+\kappa_\ell\right] + \mathcal{O}\left(\frac{1}{c^5}\right)$$

GW flux at infinity

$$\mathcal{L} = \sum_{\ell=2}^{\infty} \frac{G}{c^{2\ell+1}} \left\{ \frac{(\ell+1)(\ell+2)}{(\ell-1)\ell\ell!(2\ell+1)!!} U_L^{(1)} U_L^{(1)} + \frac{4\ell(\ell+2)}{c^2(\ell-1)(\ell+1)!(2\ell+1)!!} V_L^{(1)} V_L^{(1)} \right\}$$

Once the orbital motion of the binary is known, one can compute the GW flux *F* at infinity. This flux should be equal to the dissipation of the binary orbital energy obtained by the conservative part of the EOM or directly from the Hamiltonian.

$$\frac{dE}{dt} = -\mathcal{L}$$

From the expression of E in terms of the orbital frequency, we obtain the evolution equation of the orbital frequency, thereby, GW phase evolution.

summary:

	No spin	Spin-Linear	Spin-Squared	Tidal
Conservative dynamics	4PN	3.5PN	3PN	7PN
Energy flux at infinity	3.5PN	4PN	2PN	6PN
Radiation Reaction force	4.5PN	4PN	4.5PN	6PN
Waveform Phase (*)	3.5PN	4PN	2PN	6PN
Waveform Amplitude(*)	3PN	2PN	2PN	6PN
Black Hole Horizon Energy Flux (+)	5PN	3.5PN	4PN	-

(*): quasi-circular orbit only.
(+): with respect to the leading order luminosity.

As of 2015 April.

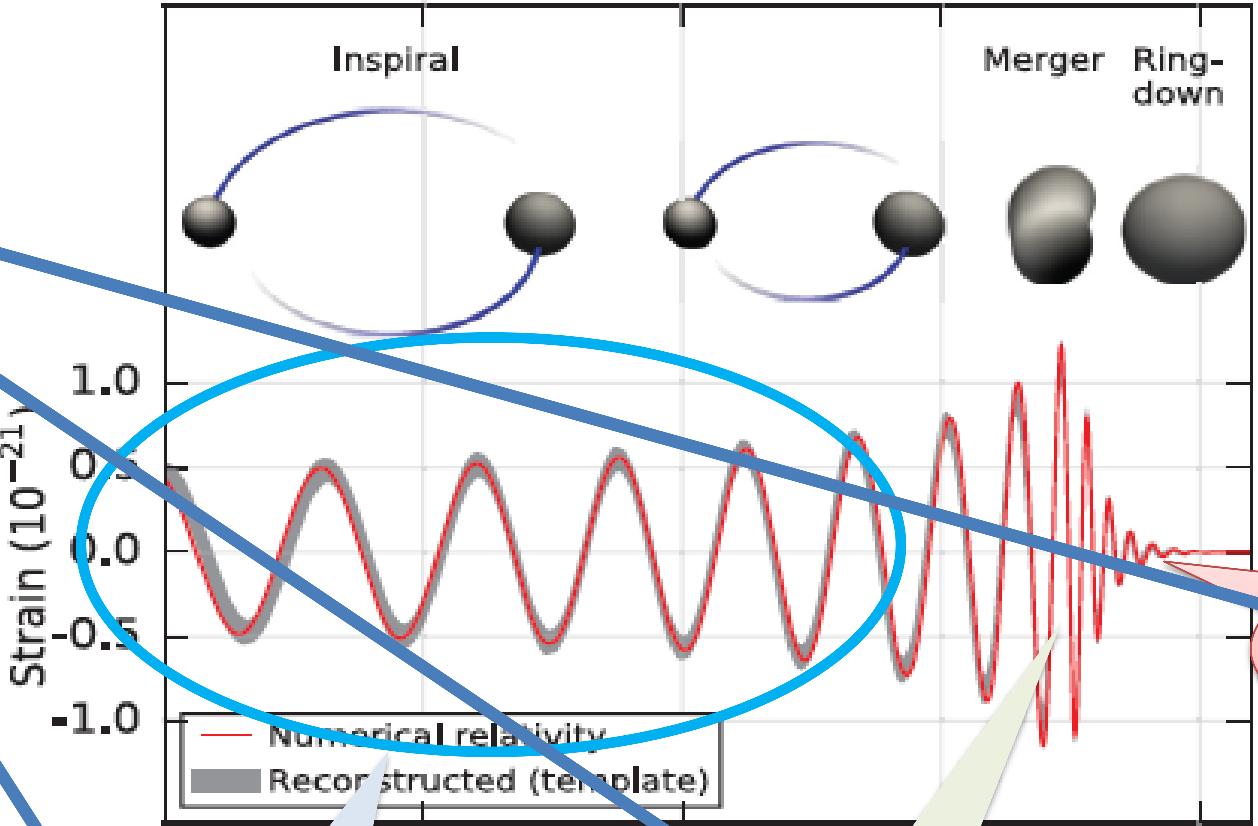
A. Buonnano & B. S. Sathyaprakash, arxiv:1410.7832

EFFECTIVE ONE-BODY APPROACH

PNA, Numerical relativity, Single star/BH Perturbation

LIGO Scientific Collaboration & Virgo collaboration (2016/02, PRL)

Combined all three
- ✓ SEOBNRv4
- ✓ IMRPhenomP



BH Perturbation

Post-Newtonian

Numerical relativity

EOB approach

Find a correspondence between

- Real problem where two-body with m1 and m2 orbiting around each other
- effective one-body problem where a test particle with mass $\mu = m_1 m_2/(m_1+m_2)$ moving in space-time endowed with an "effective metric".

EOB approach

- Obtain a relative Hamiltonian of two body problem with masses m_1 and m_2 up to some PN order. E.g., at the Newtonian order [$\mu = m_1 m_2/(m_1+m_2)$] :

$$H_N = \frac{p^2}{2\mu} - \frac{GM\mu}{r}$$

- Compute the action variables for the real problem.

$$I_i^{real} = \oint p_i dq_i$$

EOB approach

- Assume an effective metric. For non-spinning particle, assume a spherically symmetric space-time:

$$ds^2_{eff} = -A(r_{eff})dt^2 + \frac{D(r_{eff})}{A(r_{eff})}dr^2 + r^2_{eff}d\Omega^2$$

with expansions with free parameters $a_i(\nu)$ & $d_i(\nu)$ (but at the lowest order it is assumed to be the Schwarzschild metric.)

$$A(r) = 1 - \frac{2M}{r} + a_2\left(\frac{M}{r}\right)^2 + \cdots$$
$$D(r) = 1 + d_1\left(\frac{M}{r}\right) + d_2\left(\frac{M}{r}\right)^2 + \cdots$$

- $\nu = m_1 m_2/(m_1+m_2)^2$

EOB approach

- Assume one-to-one correspondence between an effective Hamiltonian (energy) and the real problem Hamiltonian (energy), specifically in the form,

$$E_{eff} = E_{real}\left(1 + \alpha_1 \frac{E_{real}}{\mu} + \cdots\right)$$

with free parameters $\alpha_i(\nu)$.

- Using the effective metric and effective energy, compute the action variables in the effective problem I_k^{eff}.
- Determine the parameters a_i, d_i, α_i from $I_k^{eff} = I_k^{real}$.
- Note that the coordinates used can be (and indeed are) different in the two problems. The correspondence is made using a canonical transformation (which depends on another set of parameters).
- Given $\alpha_i(\nu)$., we obtain the effective Hamiltonian that governs EOB problem.

EOB approach

- But actually at the 3 PN order, one needs an additional term in the effective problem (hence it is not geodesic anymore):

$$0 = \mu^2 + g_{eff}^{\alpha\beta}(x)p_\alpha p_\beta + A^{\alpha\beta\gamma\delta}(x)p_\alpha p_\beta p_\gamma p_\delta$$

- In any case, one could obtain an effective Hamiltonian.
- Furthermore, one uses Padé approximants so that the effective metric smoothly approaches the Schwarzschild one in the test particle limit ($\nu \rightarrow 0$).

Padé-approximant

- Padé–approximant of (k,m)-type where k+m = n for a series f(x) = $a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$ (a_0 is non zero) is

$$P_m^k(f(x)) = \frac{N_k(x)}{D_m(x)}$$

where N_k and D_m are polynomials of order k and m, respectively and the Taylor expansion of N_k / D_m coincides with the f(x) up to the order n.

EOB approach

- But actually at the 3 PN order, one needs an additional term in the effective problem:

$$0 = \mu^2 + g_{eff}^{\alpha\beta}(x) p_\alpha p_\beta + A^{\alpha\beta\gamma\delta}(x) p_\alpha p_\beta p_\gamma p_\delta$$

- In any case, one could obtain an effective Hamiltonian.
- Furthermore, one uses Pade approximants so that the effective metric smoothly approaches the Schwarzschild one in the test particle limit.
- The effective Hamiltonian governs the conservative part of the orbital motion. One augments it with radiation reaction forces derived from PN approach and self-force approach.

EOB approach

- Assuming quasi-circular motion governed by the effective Hamiltonian + radiation reaction force, one obtains waveform.
- Again, one introduces three sets of parameters.
  - Just before the plunge, one finds it better to introduce in the waveform a non-quasi-circular (NQC) correction term that depends on a set of parameters.
  - Also one introduces another set of parameters (in the amplitude/phase at each *l,m* mode and the time of matching) with which we can match the EOB waveform to the numerical relativity waveform for a set of parameters (mass ratio) for which NR simulations are performed.
- The resulting waveform is called EOBNR waveform.

EOB approach

- The resulting waveform is for non-spinning particles.
- Because the precession time scale is much longer than the orbital time-scale, one constructs a waveform for precessing binary by (1) introducing post-Newtonian aligned spin waveform to the EOB formalism and (2) assuming that precession waveform is equivalent to the non-precession waveform instantaneously.
- Finally match the so-obtained inspiral-plunge waveform to the ring-down waveform. This is called SEOBNR waveform.

$$h_{\ell m}^{\mathrm{EOB}}(t) = h_{\ell m}^{\text{inspiral-plunge}}(t)\theta(t_{\mathrm{match}}^{\ell m} - t) + h_{\ell m}^{\text{merger-RD}}(t)\theta(t - t_{\mathrm{match}}^{\ell m}).$$

EOB approach

- One could also incorporate tidal effects by assuming the effective metric depends on the tidal Love numbers [Bernuzzi et al., PRL **114**, 161103 2015]. The resulting waveform is $\mathrm{TEOB}_{\mathrm{Resum}}$.

$$A_T^{(+)}(u;\nu) \equiv -\sum_{\ell=2}^{4}[\kappa_A^{(\ell)} u^{2\ell+2} \hat{A}_A^{(\ell^+)} + (A \leftrightarrow B)],$$